Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach

More notes and errata for the 3rd edition

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Mathematical errors

Page 18 Third line from bottom: $k > j+1$ should be $k > j-1$ and $k < j+1$ should be $k < j-1$. In the next two lines: In the subscripts, $j+1$ should be $j-1$.

Page 19 First line: the subscript $j+2$ should be $j-2$, and $j+3$ should be $j-3$.

Page 41 The vectors given for exercise 1.1.5 should be replaced, for instance by the vectors in $\mathbb{R}^n$ $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \\ \vdots \\ n-1 \\ n \end{bmatrix}$

Page 185 Definition 2.4.5 of linear independence does not allow for the case $k = n$. It should be

Definition 2.4.5 (Linear independence). The vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ in $\mathbb{R}^n$ are *linearly independent* if, when a vector $\vec{\mathbf{w}} \in \mathbb{R}^n$ can be written as a linear combination of $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$, it can be done so in only one way:

$$\sum_{i=1}^{k} x_i \vec{\mathbf{v}}_i = \sum_{i=1}^{k} y_i \vec{\mathbf{v}}_i \quad \text{implies} \quad x_1 = y_1,\ x_2 = y_2, \dots, x_k = y_k.$$

The vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2 \in \mathbb{R}^3$ are linearly independent: there are vectors in $\mathbb{R}^3$ that cannot be written as a linear combination of $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$, but if a vector in $\mathbb{R}^3$ can be written as a linear combination of those two vectors, there is only only one way to do it.

Page 240 The last margin note should be deleted, and the second equation in example 2.8.8 should be $D_2(D_1 f)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = D_2 \underbrace{(2 + y^3)}_{D_1 f} = 3y^2$. We could define f using a, b, c rather than x, y, z, but once we have defined it using x, y, z, the partial derivatives need to be computed using those variables. Then we could *evaluate* the partial derivatives at $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to get $3b^2$.

PAGE 303 In exercise 3.1.4, the half-axes should be $\mathbb{R}^+_z$, $\mathbb{R}^+_x$, $\mathbb{R}^+_y$, $\mathbb{R}^-_z$, $\mathbb{R}^-_x$, $\mathbb{R}^-_y$

PAGE 313 Exercise 3.2.11, part d: "if A is invertible" should be "if A is orthogonal".

PAGE 347 Equation 3.6.8: $D^i D^j$ should be $D_i D_j$.

PAGE 403–404 There is an error in the proof of proposition 4.1.23. Here is a corrected proof:

By proposition 4.1.19, this is true if A is a parallelogram, in particular if A is a cube $C \in \mathcal{D}_N$. Assume A is a subset of $\mathbb{R}^n$ whose volume is well defined. This means that $\mathbf{1}_A$ is integrable, or, equivalently, that

$$\lim_{N\to\infty} \sum_{C\in\mathcal{D}_N, C\subset A} \mathrm{vol}_n(C) = \lim_{N\to\infty} \sum_{C\in\mathcal{D}_N, C\cap A\neq\emptyset} \mathrm{vol}_n(C),$$

and that the common limit is $\mathrm{vol}_n(A)$.

Since

$$\bigcup_{C\in\mathcal{D}_N, C\subset A} tC \subset tA \subset \bigcup_{C\in\mathcal{D}_N, C\cap A\neq\emptyset} tC,$$

and since $\mathrm{vol}_n(tC) = t^n \mathrm{vol}_n(C)$ for every cube, this gives

$$t^n \sum_{C\in\mathcal{D}_N, C\subset A} \mathrm{vol}_n(C) = \int \sum_{C\in\mathcal{D}_N, C\subset A} \mathbf{1}_{tC} \le L(\mathbf{1}_{tA}) \le U(\mathbf{1}_{tA}) \le \int \sum_{C\in\mathcal{D}_N, C\cap A\neq\emptyset} \mathbf{1}_{tC}$$
$$= t^n \sum_{C\in\mathcal{D}_N, C\cap A\neq\emptyset} \mathrm{vol}_n(C).$$

Since the outer terms have a common limit as $N \to \infty$, it follows that $L(\mathbf{1}_{tA}) = U(\mathbf{1}_{tA})$, and that

$$\mathrm{vol}_n(tA) = \int \mathbf{1}_{tA}(\mathbf{x})|d^n\mathbf{x}| = t^n \mathrm{vol}_n(A). \qquad \square$$

PAGE 473 For proposition 4.8.22 to be true, we must allow the matrix P to be *unitary*, the complex generalization of orthogonal. This requires complex inner product, which we have not defined. They aren't really that much more difficult than real inner products, but there is a complex conjugate which shows up and makes all the computations more complicated; in our experience students find them a lot more difficult to deal with. So for pedagogical reasons, we omitted them. We have replaced the statement as follows:

Proposition 4.8.22. *If A is an $n \times n$ complex matrix, then there exists an invertible matrix P such that $P^{-1}AP$ is upper triangular. Equivalently, there is a basis $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$ such that the matrix of A in that basis is upper triangular.*

PAGE 543 Equation 5.4.5: $K(\mathbf{x})$ should be $|K(\mathbf{x})|$.

PAGE 585 In proposition 6.4.7, "with U connected" should be "with $U - X$ connected, where X is as in definition 5.2.3." With the relaxed definition of parametrization 5.2.3, U might be connected, but $U - X$ might not. In that case checking at a single point will only give "orientation preserving" in the set of points that can be connected to that point in $U - X$.

PAGE 608 Equations 6.6.6 and 6.6.7: $\vec{\mathbf{v}}_k$, not $\vec{\mathbf{v}}_n$

PAGE 643 Equation 6.9.54 is wrong. It should be

$$f(t, \mathbf{x}) = g(\mathbf{x} \cdot \mathbf{v} - at)$$

PAGE 646 Exercise 6.9.6: The equation should be

$$f(t, \mathbf{x}) = g(\mathbf{x} \cdot \mathbf{v} - at)$$

PAGE 687 In proposition A5.1, we had defined $\mathbf{f}$ on the closed set $\overline{U}$, so differentiability doesn't really make sense. The proposition should read:

Proposition A5.1. *Let $U \subset \mathbb{R}^n$ be an open ball, $V \subset \mathbb{R}^n$ a neighborhood of $\overline{U}$, and $\mathbf{f} : V \to \mathbb{R}^m$ a differentiable mapping whose derivative satisfies the Lipschitz condition*

$$\big|[\mathbf{Df}(\mathbf{x})] - [\mathbf{Df}(\mathbf{y})]\big| \le M|\mathbf{x} - \mathbf{y}| \text{ for all } \mathbf{x}, \mathbf{y} \in \overline{U}. \qquad A5.10$$

Then for $\mathbf{x}, \mathbf{y} \in \overline{U}$,

$$\Big|\underbrace{\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})}_{\text{increment to } \mathbf{f}} - \underbrace{[\mathbf{Df}(\mathbf{x})](\mathbf{y} - \mathbf{x})}_{\substack{\text{linear approx.} \\ \text{of increment to } \mathbf{f}}}\Big| \le \frac{M}{2}|\mathbf{y} - \mathbf{x}|^2. \qquad A5.11$$

PAGE 691 First margin note: $\vec{\mathbf{h}}_0$ should be $|\vec{\mathbf{h}}_0|$ and $2\vec{\mathbf{h}}_0$ should be $2|\vec{\mathbf{h}}_0|$.

PAGE 692 Exercise A5.1: In the second displayed equation, $(\alpha - 1)$ should be $(\alpha + 1)$.

PAGE 695 Equation A7.8 should have absolute values:

$$|\vec{\mathbf{r}}(\vec{\mathbf{k}})| = |\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}}) - \mathbf{g}(\mathbf{y}_0)| \le 2|L^{-1}|\,|\mathbf{y}_0 + \vec{\mathbf{k}} - \mathbf{y}_0)|.$$

There are more serious problems with the proof of the inverse function theorem. Below is a new version.

We will begin by proving that $\mathbf{f}$ is injective on W_0, which will prove that $\mathbf{g}$ is unique. To show that $\mathbf{f}$ is injective on W_0, we will show that the function

$$\mathbf{F}(\mathbf{z}) \stackrel{\text{def}}{=} \frac{1}{2R|L^{-1}|} L^{-1}\Big(\mathbf{f}\big(\mathbf{x}_0 + 2R|L^{-1}|\mathbf{z}\big) - \mathbf{y}_0\Big)$$

satisfies the hypotheses of lemma A1.

The function $\mathbf{F}$ is designed so that it is defined in the unit ball of $\mathbb{R}^n$, with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and $[\mathbf{DF}(\mathbf{0})] = I$. Let us check that the Lipschitz condition on $\mathbf{f}$ (see equation 2.10.12) translates into the simpler Lipschitz condition of lemma A1:

$$|[\mathbf{DF}(\mathbf{z}_1)] - [\mathbf{DF}(\mathbf{z}_2)]| = \Big|L^{-1}\Big[\mathbf{Df}\Big(\mathbf{x}_0 + 2R|L^{-1}|\mathbf{z}_1\Big)\Big] - L^{-1}\Big[\mathbf{Df}\Big(\mathbf{x}_0 + 2R|L^{-1}|\mathbf{z}_2\Big)\Big]\Big|$$
$$\leq |L^{-1}|\frac{1}{2R|L^{-1}|^2}\Big|2R|L^{-1}|\mathbf{z}_1 - 2R|L^{-1}|\mathbf{z}_2\Big| = |\mathbf{z}_1 - \mathbf{z}_2|.$$

Lemma A1. *Let B be the unit ball of $\mathbb{R}^n$, and let $\mathbf{F} : B \to \mathbb{R}^n$ be a C^1 mapping such that*

$$\mathbf{F}(\mathbf{0}) = \mathbf{0}, \quad [\mathbf{DF}(\mathbf{0})] = I, \quad \text{and} \quad \big|[\mathbf{DF}(\mathbf{x})] - [\mathbf{DF}(\mathbf{y})]\big| \leq |\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in B$. Then $\mathbf{F}$ is injective.

Proof. Using corollary 1.9.2, we can write

$$\begin{aligned}
|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})| &= \big|(\mathbf{x}-\mathbf{y}) + \big(\mathbf{F}(\mathbf{x}) - \mathbf{x}) - (\mathbf{F}(\mathbf{y}) - \mathbf{y})\big)\big| \\
&= \big|(\mathbf{x}-\mathbf{y}) + \big(\mathbf{F} - I)(\mathbf{x}) - (\mathbf{F} - I)(\mathbf{y})\big| \\
&\geq |\mathbf{x}-\mathbf{y}| - \sup_{\mathbf{z}\in[\mathbf{x},\mathbf{y}]}\Big|[\mathbf{DF}(\mathbf{z})] - I\Big|\,|\mathbf{x}-\mathbf{y}| \\
&= |\mathbf{x}-\mathbf{y}| - \sup_{\mathbf{z}\in[\mathbf{x},\mathbf{y}]}\Big|[\mathbf{DF}(\mathbf{z})] - [\mathbf{DF}(\mathbf{0})]\Big|\,|\mathbf{x}-\mathbf{y}| \\
&\geq |\mathbf{x}-\mathbf{y}| - \sup_{\mathbf{z}\in[\mathbf{x},\mathbf{y}]}\Big|\mathbf{z} - \mathbf{0}\Big|\,|\mathbf{x}-\mathbf{y}| \\
&= |\mathbf{x}-\mathbf{y}|\big(1 - \sup(|\mathbf{x}|, |\mathbf{y}|)\big).
\end{aligned}$$

(To go from the next-to-last line to the last line, we use that $\sup_{\mathbf{z}\in[\mathbf{x},\mathbf{y}]}\big|\mathbf{z}\big| = \sup(|\mathbf{x}|, |\mathbf{y}|)$, since the point of a line segment farthest from the origin is always one of the endpoints.) Thus $\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$. $\square$

It follows that $\mathbf{g}$ is the unique map $V \to W_0$ such that $\mathbf{f} \circ \mathbf{g}$ is the identity on V. If $\mathbf{g}_1$ is such a map, then $\mathbf{g}_1(\mathbf{y})$ is an inverse image of $\mathbf{y}$ under $\mathbf{f}$ that is an element of W_0, and there is at most one (hence exactly one) such inverse image, namely $\mathbf{g}(\mathbf{y})$.

Proving that g is continuous on V.

The inequality in the proof of lemma A1 gives us a bit more, it tells us that $\mathbf{g}$ is continuous, and even gives a modulus of continuity. Indeed, for any $\mathbf{x}_1, \mathbf{x}_2 \in W_0$ we have

$$\begin{aligned}
|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)| &= 2R|L^{-1}|\left|L\left(\mathbf{F}\left(\frac{\mathbf{x}_1 - \mathbf{x}_0}{2R|L^{-1}|}\right) - \mathbf{F}\left(\frac{\mathbf{x}_2 - \mathbf{x}_0}{2R|L^{-1}|}\right)\right)\right| \\
&\geq 2R\left|\mathbf{F}\left(\frac{\mathbf{x}_1 - \mathbf{x}_0}{2R|L^{-1}|}\right) - \mathbf{F}\left(\frac{\mathbf{x}_2 - \mathbf{x}_0}{2R|L^{-1}|}\right)\right| \\
&\geq 2R\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{2R|L^{-1}|}\left(1 - \frac{\sup(|\mathbf{x}_1 - \mathbf{x}_0|, |\mathbf{x}_2 - \mathbf{x}_0|)}{2R|L^{-1}|}\right) \\
&= \frac{1}{|L^{-1}|}\left(1 - \frac{\sup(|\mathbf{x}_1 - \mathbf{x}_0|, |\mathbf{x}_2 - \mathbf{x}_0|)}{2R|L^{-1}|}\right)|\mathbf{x}_1 - \mathbf{x}_2|.
\end{aligned}$$

Thus if $\mathbf{x}_1 = \mathbf{g}(\mathbf{y}_1)$ and $\mathbf{x}_2 = \mathbf{g}(\mathbf{y}_2)$, with

$$\sup(|\mathbf{x}_1 - \mathbf{x}_0|, |\mathbf{x}_2 - \mathbf{x}_0|) \leq 2R'|L^{-1}|$$

for some $R' < R$, then

$$|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)| \geq |\mathbf{x}_1 - \mathbf{x}_2|\frac{R - R'}{|L^{-1}|R}, \quad \text{i.e.,} \quad |\mathbf{g}(\mathbf{y}_1) - \mathbf{g}(\mathbf{y}_2)| \leq \frac{R|L^{-1}|}{R - R'}|\mathbf{y}_1 - \mathbf{y}_2|. \qquad (A1)$$

Changing the base point

Note that to show the existence of $R > 0$ and of $\mathbf{g}$, the only hypotheses about $\mathbf{f}$ we used were that $\mathbf{f}(\mathbf{x}_0) = \mathbf{y}_0$, that $[\mathbf{Df}(\mathbf{x}_0)]$ is invertible, and that $\mathbf{x} \mapsto [\mathbf{Df}(\mathbf{x})]$ is Lipschitz in a neighborhood of $\mathbf{x}_0$. For any points $\mathbf{y}_0' \in V$ and $\mathbf{x}_0' = \mathbf{g}(\mathbf{y}_0')$, the same hypotheses are true.

Write $\mathbf{g} = \mathbf{g}_{\mathbf{x}_0,\mathbf{y}_0}$. For any $\mathbf{y}_0' \in V$ and $\mathbf{x}_0' \stackrel{\text{def}}{=} \mathbf{g}(\mathbf{y}_0')$, there is an analogous map $\mathbf{g}_{\mathbf{x}_0',\mathbf{y}_0'}$. This map also specifies an inverse image of $\mathbf{y}$ under $\mathbf{f}$, and since $\mathbf{f}$ is injective on W_0, it must be the same inverse image in some neighborhood of $\mathbf{y}_o'$. Thus if we prove that $\mathbf{g} = \mathbf{g}_{\mathbf{x}_0,\mathbf{y}_0}$ is differentiable at $\mathbf{y}_0$, the same proof will show that $g_{\mathbf{x}_0',\mathbf{y}_0'}$ is differentiable at $\mathbf{y}_0'$, hence $\mathbf{g}$ is also differentiable at $\mathbf{y}_0'$, since it coincides with $g_{\mathbf{x}_0',\mathbf{y}_0'}$ in a neighborhood of $\mathbf{y}_0'$.

Let us see that $\mathbf{g}(V)$ is open. From the argument above, if $\mathbf{g}(V)$ contains a neighborhood of $\mathbf{x}_0$, then it will contain a neighborhood of ll $\mathbf{x}_0' \in \mathbf{g}(V)$. By the injectivity of $\mathbf{f}$ on W_0, $\mathbf{g}(V)$ does contain a neighborhood of $\mathbf{x}_0$: if $\mathbf{x}$ is sufficiently close to $\mathbf{x}_0$ then $\mathbf{f}(\mathbf{x})$ is in V so we can consider the element $\mathbf{x}_1 \stackrel{\text{def}}{=} \mathbf{g}(\mathbf{f}(\mathbf{x})$ of W_0, and

$$\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{f}(\mathbf{x})$$

which by injectivity of $\mathbf{f}$ implies $\mathbf{x} = \mathbf{x}_1 \in \mathbf{g}(V)$.

Proving that $\mathbf{g}$ is differentiable at $\mathbf{y}_0$

Here we show that $\mathbf{g}$ is differentiable at $\mathbf{y}_0$, with derivative $[\mathbf{Dg}(\mathbf{y}_0)] = L^{-1}$, i.e., that

$$\lim_{\vec{\mathbf{k}} \to \vec{\mathbf{0}}} \frac{\big(\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}}) - \mathbf{g}(\mathbf{y}_0)\big) - L^{-1}\vec{\mathbf{k}}}{|\vec{\mathbf{k}}|} = \vec{\mathbf{0}}. \qquad (A2)$$

When $|\mathbf{y}_0 + \vec{\mathbf{k}}| \in V$, define $\vec{\mathbf{r}}(\vec{\mathbf{k}})$ to be the increment to $\mathbf{x}_0$ that under $\mathbf{f}$ gives the increment $\vec{\mathbf{k}}$ to $\mathbf{y}_0$:

$$\mathbf{f}\big(\mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}})\big) = \mathbf{y}_0 + \vec{\mathbf{k}}, \qquad (A3)$$

or, equivalently,

$$\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}}) = \mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}}). \qquad (A4)$$

Substitute the right side of equation (A4) for $\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}})$ in the left side of equation (A2), remembering that $\mathbf{g}(\mathbf{y}_0) = \mathbf{x}_0$. This gives find

$$\lim_{\vec{\mathbf{k}}\to\vec{\mathbf{0}}} \frac{\mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}}) - \mathbf{x}_0 - L^{-1}\vec{\mathbf{k}}}{|\vec{\mathbf{k}}|} \;=\; \lim_{\vec{\mathbf{k}}\to\vec{\mathbf{0}}} \frac{\vec{\mathbf{r}}(\vec{\mathbf{k}}) - L^{-1}\vec{\mathbf{k}}}{|\vec{\mathbf{k}}|}\frac{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}$$

$$= \lim_{\vec{\mathbf{k}}\to\vec{\mathbf{0}}} \frac{L^{-1}\left(L\vec{\mathbf{r}}(\vec{\mathbf{k}}) - \overbrace{\Big(\mathbf{f}\big(\mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}})\big) - \mathbf{f}(\mathbf{x}_0)\Big)}^{\vec{\mathbf{k}} \text{ by equation (A3)}}\right)}{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}\,\frac{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}{|\vec{\mathbf{k}}|}.$$

Since $\mathbf{f}$ is differentiable at $\mathbf{x}_0$, the term

$$\frac{L\vec{\mathbf{r}}(\vec{\mathbf{k}}) - \mathbf{f}(\mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}})) + \mathbf{f}(\mathbf{x}_0)}{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}$$

has limit $\vec{\mathbf{0}}$ as $\vec{\mathbf{r}}(\vec{\mathbf{k}}) \to \vec{\mathbf{0}}$. The differentiability of $\mathbf{g}$ at $\mathbf{y}_0$ (equation A7.3) will follow from part 5 of theorem 1.5.23 if we show that $\vec{\mathbf{r}}(\vec{\mathbf{k}}) \to \vec{\mathbf{0}}$ when $\vec{\mathbf{k}} \to \vec{\mathbf{0}}$, fast enough that $|\vec{\mathbf{r}}(\vec{\mathbf{k}})|/|\vec{\mathbf{k}}|$ remains bounded. Equation (A4) tells us that

$$|\vec{\mathbf{r}}(\vec{\mathbf{k}})| = \; |\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}}) - \mathbf{g}(\mathbf{y}_0)|$$

and since $\mathbf{g}$ is continuous, $|\vec{\mathbf{r}}(\vec{\mathbf{k}})|$ does tend to 0 with $|\vec{\mathbf{k}}|$. The proof that $\mathbf{g}$ is continuous (equation (A1)) shows more. Let $\mathbf{x} = \mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}})$; as $\vec{\mathbf{k}}$ tends to $\vec{\mathbf{0}}$, the length $|\mathbf{x} - \mathbf{x}_0|$ also tends to 0 since $\mathbf{g}$ is continuous, in particular it will satisfy $|\mathbf{x} - \mathbf{x}_0| < RL^{-1}$ when $|\vec{\mathbf{k}}|$ is sufficiently small. Equation (A1) says that for these values of $\vec{\mathbf{k}}$ we have

$$\frac{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}{|\vec{\mathbf{k}}|} \le \frac{2R|L^{-1}|}{R - R/2} = 4|L^{-1}|$$

so $|\vec{\mathbf{r}}(\vec{\mathbf{k}})|/|\vec{\mathbf{k}}|$ remains bounded as $\vec{\mathbf{k}} \to \vec{\mathbf{0}}$.

Proving that g is continuously differentiable on V. This follows immediately from the formula $[\mathbf{Dg}(\mathbf{y})] = [\mathbf{Df}(\mathbf{g}(\mathbf{y}))]^{-1}$, derived from the chain rule.

Proving equation 2.10.14 Suppose $|\mathbf{x} - \mathbf{x}_0| < R_1$. Then

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| \;\le\; |\mathbf{x} - \mathbf{x}_0| \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big| \;<\; R_1 \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big|. \qquad (A5)$$

So if $R_1 \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big| < R$, then $\mathbf{f}(\mathbf{x})$ is in V. We find a bound for $|[\mathbf{Df}(\mathbf{z})]|$, when $|\mathbf{z} - \mathbf{x}_0| < R_1$:

$$|[\mathbf{Df}(\mathbf{z})] - [\mathbf{Df}(\mathbf{x}_0)]| = |[\mathbf{Df}(\mathbf{z})] - L| \underbrace{\le}_{\text{Eq. 2.10.12}} \frac{1}{2R|L^{-1}|^2}\,|\mathbf{z} - \mathbf{x}_0| < \frac{R_1}{2R|L^{-1}|^2}$$

so

$$|[\mathbf{Df}(\mathbf{z})]| \le |L| + \frac{R_1}{2R|L^{-1}|^2}, \text{ i.e., } \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big| \le |L| + \frac{R_1}{2R|L^{-1}|^2}.$$

Substituting this bound for $R_1 \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big|$ in equation (A5), we see that if

$$R \;\ge\; \left(|L| + \frac{R_1}{2R|L^{-1}|^2}\right) R_1, \qquad (A6)$$

then $|\mathbf{x}-\mathbf{x}_0| < R_1$ implies $\mathbf{f}(\mathbf{x}) \in V$. Then $\mathbf{g}(\mathbf{f}(\mathbf{x}))$ is an inverse image of $\mathbf{f}(\mathbf{x})$ in W_0, but since $\mathbf{f}$ is injective on W_0 there only is one, so $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$ and $\mathbf{x} \in \mathbf{g}(V)$.

We leave it to the reader to check that the largest R_1 satisfying equation (A6) is

$$R_1 = R|L^{-1}|^2 \left(-|L| + \sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}} \right). \qquad A7.15$$

PAGE 733 Since proposition 4.8.22 has been changed, the proof needs to be changed:

Proof. The proof is by induction on n. It is obvious if $n = 1$, so suppose $n \geq 2$ and assume the result for all $(n-1) \times (n-1)$ matrices.

Find an eigenvector $\vec{\mathbf{v}}_1$ with eigenvalue λ_1 (which exists by the fundamental theorem of algebra and the procedure described in section 2.7). Choose vectors $\vec{\mathbf{w}}_2, \dots, \vec{\mathbf{w}}_n$ such that $\vec{\mathbf{v}}_1, \vec{\mathbf{w}}_2, \dots, \vec{\mathbf{w}}_n$ is a basis of $\mathbb{R}^n$. Then $T \stackrel{\text{def}}{=} [\vec{\mathbf{v}}_1, \vec{\mathbf{w}}_2, \dots, \vec{\mathbf{w}}_n]$ is invertible, and since $\vec{\mathbf{v}}_1$ is an eigenvector of A, the first column of $B \stackrel{\text{def}}{=} T^{-1}AT$ is $\lambda_1\vec{\mathbf{e}}_1$, i.e., we can write

$$B \stackrel{\text{def}}{=} T^{-1}AT = \left[\begin{array}{c|c} \lambda_1 & \beta \\ \hline 0 & \\ \vdots & \widetilde{B} \\ 0 & \end{array}\right].$$

where β is some $1 \times (n-1)$ matrix, and $\widetilde{B}$ is an $(n-1) \times (n-1)$ matrix.

By our inductive hypothesis, we can find an invertible matrix $\widetilde{Q}$ such that $\widetilde{Q}^{-1}\widetilde{B}\widetilde{Q}$ is upper triangular. Set

$$Q = \left[\begin{array}{c|c} 1 & \dots \ 0 \ \dots \\ \hline 0 & \\ \vdots & \widetilde{B} \\ 0 & \end{array}\right] \quad \text{so that} \quad Q^{-1}BQ = \left[\begin{array}{c|c} \lambda_1 & \beta\widetilde{Q} \\ \hline 0 & \\ \vdots & \widetilde{Q}^{-1}\widetilde{B}\widetilde{Q} \\ 0 & \end{array}\right]. \qquad A18.17$$

In particular, $Q^{-1}BQ$ is upper triangular. Set $P = TQ$, then

$$P^1AP = Q^{-1}T^{-1}ATQ = Q^{-1}BQ \qquad A18.18$$

is upper triangular. □

PAGE 734 Corollary A18.1 does not follow from the new version of proposition 4.8.22.

PAGES 738–740 Because we no longer can use corollary A18.1, some changes are required in the subsection "completing the proof of the change of variables formula". At the bottom of page 738, "we will denote by Z the union of these cubes" should become "We will denote by Z the union of the closure of these cubes".

The top of page 739 (up to Lemma A19.6) should become

We will also require that N_1 be large enough so that $Z \subset U$. Then $X \cup Z$ is a compact subset of U. Call M the Lipschitz ratio of $[\mathbf{D}\Phi]$, and set

$$K = \sup_{\mathbf{x} \in X \cup Z} |\det[\mathbf{D}\Phi(\mathbf{x})]| \quad \text{and} \quad L = \sup_{\mathbf{y} \in Y} |f(\mathbf{y})|. \qquad A19.25$$

We know that K is well defined because $|\det[\mathbf{D}\Phi(\mathbf{x})]|$ is continuous on $X \cup Z$.

In the remainder of the proof, K^n should be replaced by K.

Other errors and miscellaneous notes

PAGE 77 A trick for remembering how to compute the cross product is to write

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \det \begin{bmatrix} \vec{\mathbf{e}}_1 & a_1 & b_1 \\ \vec{\mathbf{e}}_2 & a_2 & b_2 \\ \vec{\mathbf{e}}_3 & a_3 & b_3 \end{bmatrix}.$$

It isn't clear what meaning to attach to a matrix with some entries vectors and some scalars, but if you just treat each entry as something to be multiplied, it gives the right answer.

PAGE 99 Second paragraph: A monomial function is a function $\mathbb{R}^n \to \mathbb{R}$.

PAGE 100 In figure 1.5.7, some of the points are mislabeled. The vector $\mathbf{a}_3$ (which should be written $\vec{\mathbf{a}}_3$) goes from $\mathbf{s}_2$ to $\mathbf{s}_3$, and so forth.

PAGE 111 Proof of theorem 1.6.9, the statement at the end of the first paragraph, that if $|f(\mathbf{x}) - f(\mathbf{b})| < \epsilon$, then $|f(\mathbf{x})| < |f(\mathbf{b})| + \epsilon$, is justified by the triangle inequality:

$$|f(\mathbf{x})| = |f(\mathbf{x}) - f(\mathbf{b}) + f(\mathbf{b})| \ \leq\ |f(\mathbf{x}) - f(\mathbf{b})| + |f(\mathbf{b})| \ <\ \epsilon + |f(\mathbf{b})|.$$

PAGE 122 In the first margin note, the word "maneuverability" is misspelled.

PAGE 128 Since the gradient is a (column) vector, it would be better to use square brackets in equation 1.7.2:

$$\operatorname{grad} f(\mathbf{a}) = \vec{\nabla} f(\mathbf{a}) = \begin{bmatrix} D_1 f(\mathbf{a}) \\ \vdots \\ D_n f(\mathbf{a}) \end{bmatrix}.$$

PAGE 138 In the third margin note, it might be clearer to write "By theorem 1.9.7, requiring that the first partials be differentiable is weaker than requiring that the second partials be continuous."

PAGE 148 In equation 1.9.5, $(\mathbf{b} - \mathbf{a})$ should have an arrow over it both places it appears.

PAGE 266 Three lines after equation 2.10.17, $\mathbf{x}_1 = \mathbf{x}_0 + \vec{\mathbf{h}}_0(\mathbf{y})$ should be

$$\mathbf{x}_1(\mathbf{y}) = \mathbf{x}_0 + \vec{\mathbf{h}}_0(\mathbf{y}).$$

PAGE 327 Line immediately after equation 3.4.3: $\pi \approx 3.1416$, not $\pi \approx 3.1415$.

PAGE 334 We have changed definition 3.5.1 to say that a quadratic form is a polynomial *function* :

A quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function in the variables $x_1, \dots, x_n$, all of whose terms are of degree 2.

PAGE 338 We have changed the sentence after equation 3.5.20 to read

Only the second decomposition reflects theorem 3.5.3. In the first, the three functions $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto y$, and $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x + y$ are linearly dependent.

Page 352 In the line immediately before equation 3.7.8, F should be $\mathbf{F}$.

Page 355 Two lines before equation 3.7.12, "line on that line" should be "lie on that line".

Page 399 Second line. "With support in $[a, b]$" means $f(x) = 0$ if $x \notin [a, b]$. If $f(x) \neq 0$ for x in some other interval, say $[c, d]$, we would say "with support in $[a, b] \cup [c, d]$".

Page 409 Definition 4.2.4 defines probability measure using the undefined notion of probability. Here is a more precise definition:

Definition 4.2.4 (Probability measure). The *probability measure* $\mathbf{P}$ takes an event $A \subset S$ and returns a number $\mathbf{P}(A) \in [0, 1]$ such that (1) $\mathbf{P}(S) = 1$, and (2) if $\mathbf{P}(A \cap B) = \phi$ for $A, B \subset S$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$.

We say that $\mathbf{P}(A)$ corresponds to the *probability* of an outcome of the experiment being in A; it can range from 0 (it is certain that the outcome will not be in A) to 1 (it is certain that it will be in A).

Page 608 Two lines before equation 6.6.6: *an* $(n - k) \times n$ matrix

Page 637 First line after equation 6.9.21: the constant of proportionality μ_0 is known as the *permeability* of free space.

Page 642 First margin note: Alternatively, one can measure the speed of light; in SI (the international system of units), the permeability μ_0 is defined to be $4\pi/10^7$, and then equation 6.9.61 says that ϵ_0 is $1/(\mu_0 c^2)$.

Page 643 Three lines after equation 6.9.56: "in free space, where there are no charges or currents" should be simply "in the absence of charges or currents".

Page 646 Similarly, in exercises 6.9.8 and 6.9.9, "in free space (no charges or currents)" should be "in the absence of charges or currents".

Page 650 Equation 6.10.22: An end parenthesis is missing after $f(x_{i+1}$.

Page 665 Margin note: "Over an oriented curve", not over "an oriented surface".

Page 766 About one-third of the way down the page, the sentence about the remainder should read

the remainder satisfies $|R(f)(\vec{\mathbf{x}})| \leq C|\vec{\mathbf{x}}|^2$, for some constant C.