Chapter 8

Advances in solving linear control systems

This chapter is devoted to iterative solutions of Sylvester and continuous-time algebraic Riccati equations using the *SOR-like* method, recently introduced by the author. These equations are important in many design problems appearing in control and filter theory. Section 8.1 analyzes the solution of Sylvester equations [92, 93]. Section 8.2 discusses the numerical solution of continuous-time algebraic Riccati equations [95].

8.1 Sylvester equations

The matrix equation

$$\mathbf{A}\mathbf{X} - \mathbf{X}\mathbf{B} = \mathbf{C}, \tag{8.1}$$

with $\mathbf{A} \in \mathbb{R}^{m\times m}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ and $\mathbf{C}, \mathbf{X} \in \mathbb{R}^{m\times n}$, is known as the *Sylvester equation.* It has many applications in control theory. When $\mathbf{B} = -\mathbf{A}^\top$, it reduces to the well-known *Lyapunov equation.*

Equation (8.1) is a linear equation for X, and can be written as a large linear system of the standard form for linear equations:

$$\mathbf{G}\mathbf{x} = \mathbf{c} \tag{8.2}$$

for an $mn \times mn$ matrix

$$\mathbf{G} = \mathbf{I}_n \otimes \mathbf{A} - \mathbf{B}^\top \otimes \mathbf{I}_m, \tag{8.3}$$

where $\otimes$ denotes the Kronecker product, and $\mathbf{x}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^{mn}$ whose components are the entries of successive rows of the matrices $\mathbf{X}$ and $\mathbf{C}$ respectively.

Equation (8.1) has a unique solution if and only if the matrices $\mathbf{A}$ and $\mathbf{B}$ have no common eigenvalues [20].

In the literature, several methods have been proposed for solving (8.1) [7, 24, 30, 63]. A survey of properties and applications of the Sylvester equation in control theory is presented in [15]. Recently, block Lanczos and Arnoldi methods have been developed for solving (8.1) [25]. Simoncini [50] extended the Hu-Reichel algorithm [30] to the block form, and El Guennouni et al. [25] proposed new Krylov subspace algorithms based on block Arnoldi and Lanczos methods.

In subsection 8.1.1 we present the simple *SOR-like* method for solving (8.1). Subsection 8.1.2 discusses results of numerical experiments using the SOR-like method to solve Sylvester equations obtained by means of two different separation models of elliptic partial differential equations, for examples derived from Problem E, originally used in [30]. One of these models, called the *separation model A*, is used in [30]; the second, the *separation model B*, is new.

As is demonstrated in these numerical experiments, the SOR-like method seems to be a very efficient tool, strongly competitive with Krylov subspace techniques, especially with separation model B.

8.1.1 The SOR-like method

Assume that the matrix $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = \mathbf{K} - \mathbf{L} - \mathbf{U}, \tag{8.4}$$

where $\mathbf{K}$ is nonsingular diagonal, $\mathbf{L}$ is the strictly lower triangular part of $\mathbf{A}$, and $\mathbf{U}$ is its strictly upper triangular part. Then equation (8.1) can be rewritten as follows:

$$\mathbf{KX} = \mathbf{LX} + \mathbf{UX} + \mathbf{XB} + \mathbf{C} \tag{8.5}$$

or, equivalently, as

$$\mathbf{X} = \mathbf{K^{-1}}\Big[\mathbf{LX} + \mathbf{UX} + \mathbf{XB} + \mathbf{C}\Big]. \tag{8.6}$$

The iteration process is depicted in figures 8.1.1 and 8.1.2 for the example with $m = 7$ and $n = 3$, where the nonzero entries of the matrices $\mathbf{K}$, $\mathbf{L}$, $\mathbf{U}$, $\mathbf{B}$, and $\mathbf{C}$ are denoted by "$*$"; the entries of $\mathbf{X}^{(t-1)}$ computed in the iteration $t-1$ are denoted by "$\circ$"; the entries of $\mathbf{X}^{(t)}$ computed in the iteration t are denoted by "$\bullet$"; and the entry of $\mathbf{X}^{(t)}$ currently computed is denoted by "$\circledast$".

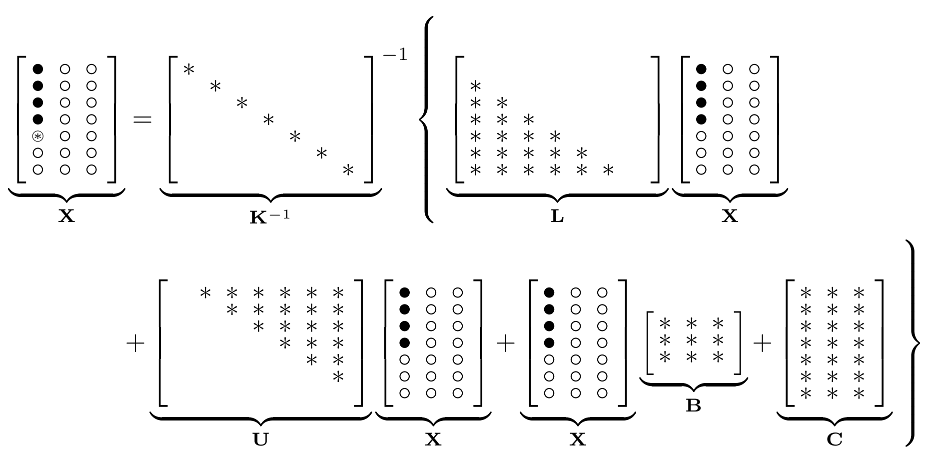

FIGURE 8.1.1

The entries of $\mathbf{X}^{(t)}$ are computed by columns, where the entries of $\mathbf{X}^{(t)}$ are used to compute the product $\mathbf{LX}$ and the entries of $\mathbf{X}^{(t-1)}$ are used to compute the product