6.14 Review exercises for Chapter 6

In Exercise 6.1, B is an $n \times n$ matrix and φ and ψ are both 1-forms on $\mathbb{R}^3$; $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are vectors in $\mathbb{R}^3$; f is integrable.

6.1 Which of the following are numbers? Identify those that are not.

a. $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$	b. $dx_1 \wedge dx_2(\vec{\mathbf{v}}, \vec{\mathbf{w}})$	c. $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$	d. $\det B$
e. $\operatorname{rank} B$	f. $\operatorname{tr} B$	g. $\dim A^k(\mathbb{R}^n)$	h. $\lvert\vec{\mathbf{v}}\rvert$
i. $A^k(\mathbb{R}^k)$	j. $\varphi \wedge \psi(\vec{\mathbf{v}}, \vec{\mathbf{w}})$	k. $\int_{\mathbb{R}} f(x)\,dx$	l. $\operatorname{sgn}(\sigma)$

6.2 Let $\vec{F}$ be a vector field in $\mathbb{R}^n$, let $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}$ be vectors in $\mathbb{R}^n$, and let φ be the $(n-1)$-form on $\mathbb{R}^n$ given by

$$\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}) = \det[\vec{F}(\mathbf{x}), \vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}].$$

Write φ as a linear combination of elementary $(n-1)$-forms on $\mathbb{R}^n$, in terms of the coordinates of $\vec{F}$.

6.3 Use Definition 6.1.12 to write the wedge product $\varphi \wedge \psi(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4)$, where φ is a 1-form and ψ is a 3-form, as a combination of values of φ and ψ evaluated on appropriate vectors (as in equations 6.1.28 and 6.1.32).

6.4 Set up each of the following integrals of form fields over parametrized domains as an ordinary multiple integral.

a. $\int_{[\gamma(I)]} y^2 dy + x^2 dz$, where $I = [0, a]$ and $\gamma(t) = \begin{pmatrix} t^3 \\ t^2+1 \\ t^2-1 \end{pmatrix}$

b. $\int_{[\gamma(U)]} \sin y^2\, dx \wedge dz$, where $U = [0,a] \times [0,b]$, and $\gamma\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^2 - v \\ uv \\ v^4 \end{pmatrix}$

6.5 Find a 1-form field on $\mathbb{R}^2$ whose sign orients the circle of radius 1 centered at $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ in the clockwise direction.

6.6 Find an orientation Ω for the surface $S \subset \mathbb{R}^3$ given by

$$x^2 + y^3 + z^4 = 1.$$

6.7 Consider the manifold $S^3 \subset \mathbb{R}^4$ of equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

a. Show that $\operatorname{sgn} dx_1 \wedge dx_2 \wedge dx_3$ is not an orientation of S^3.

b. Show that $\Omega_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3) = \operatorname{sgn}\det[\mathbf{x}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$ is an orientation.

6.8 a. Show that the locus $M \subset \mathbb{R}^4$ given by $x_1^2 + x_2^2 + x_1 x_4^2 = 1$ is a smooth manifold.

b. Find a 3-form field whose sign orients M.

6.9 In Example 6.4.3 we saw that $\gamma_1(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\gamma_2(\theta) = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$ give opposite orientations. Confirm (Proposition 6.4.8) that $\det[\mathbf{D}(\gamma_2^{-1} \circ \gamma_1)] < 0$.

6.10 Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be coordinates in $\mathbb{C}^2$. Compute the integral of $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ over the part of the locus of equation $z_2 = z_1^k$ where $|z_1| < 1$, oriented by $\operatorname{sgn} dx_1 \wedge dy_1$.

6.11 For the following 1-forms, write down the corresponding vector field. Sketch the vector field. Describe a path over which the work of the 1-form would be small. Describe a path over which the work would be large.

a. $(x^2 + y^2)\,dz$, on $\mathbb{R}^3$ b. $y\,dx - x\,dy - z\,dz$, on $\mathbb{R}^3$

6.12 a. In $\mathbb{R}^2$, a vector field defines two 1-forms: the work and the flux. Show that they are related by formula

$$W_{\vec{F}}(\vec{\mathbf{v}}) = \Phi_{\vec{F}}\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\vec{\mathbf{v}}\right).$$

b. What does the transformation $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ correspond to geometrically?

c. Can you explain why the work and the flux on $\mathbb{R}^2$ are related by the formula in part a?

$$\vec{F}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{bmatrix}y\\ -z\\ yz\end{bmatrix}$$

Vector field for Exercise 6.13.

6.13 Find the flux of the vector field $\vec{F}$ shown in the margin, through S, where S is the part of the cone $z = \sqrt{x^2+y^2}$ where $x, y \geq 0,\ x^2+y^2 \leq R$, and S is oriented by the inward-pointing normal (i.e., the flux measures flow into the cone).

6.14 Let S be the part of the surface of equation $z = \sin xy + 2$ where

$$x^2 + y^2 \leq 1 \quad \text{and} \quad x \geq 0,$$

oriented by the upward-pointing normal; let $\vec{F} = \begin{bmatrix}0\\ 0\\ x+y\end{bmatrix}$. What is the flux of $\vec{F}$ through S?

6.15 Consider the manifold $S^3 \subset \mathbb{R}^4$ of equation $x_1^2+x_2^2+x_3^2+x_4^2 = 1$, oriented by $\Omega_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3) = \operatorname{sgn}\det[\mathbf{x}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$. Let X be the subset of S^3 where $x_4 \leq 0$.

a. Show that X is a piece-with-boundary of S^3.

b. Find a basis for the tangent space $T_{\mathbf{x}}(\partial X)$ at the point $\mathbf{x} = \begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix} \in \partial X$ which is direct for the boundary orientation.

6.16 a. Compute the derivative of $xy\, dz$ from the definition.

b. Compute the same derivative using the formulas given in Theorem 6.7.4, stating clearly at each stage what property you are using.

6.17 a. Let $\varphi = xyz\, dy$. Compute from the definition the number

$$\mathbf{d}\varphi\left(P_{\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}}(\vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3)\right).$$

b. What is $\mathbf{d}\varphi$? Use your result to check the computation in part a.

Exercise 6.18 gives another way to derive equation 5.3.50.

6.18 Let $\vec{\mathbf{r}} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$ be the radial vector field in $\mathbb{R}^n$.

a. Show that $\mathbf{d}(\Phi_{\vec{\mathbf{r}}}) = n(dx_1 \wedge \cdots \wedge dx_n)$.

b. Let $B_1^n(\mathbf{0})$ and S^{n-1} be the unit ball and the unit sphere in $\mathbb{R}^n$, the ball with the standard orientation and the sphere with the boundary orientation. Use Stokes's theorem to prove

$$\operatorname{vol}_n\left(B_1^n(\mathbf{0})\right) = \frac{1}{n}\operatorname{vol}_{n-1}(S^{n-1}).$$

6.19 Using Theorem 6.8.3, prove the equations

$$\operatorname{curl}(\operatorname{grad} f) = \vec{\mathbf{0}} \quad \text{and} \quad \operatorname{div}(\operatorname{curl} \vec{F}) = 0$$

for any function f and any vector field $\vec{F}$ (of class at least C^2).

6.20 a. For what vector field $\vec{F}$ is the 1-form on $\mathbb{R}^3$

$$x^2\, dx + y^2 z\, dy + xy\, dz \quad \text{the work form field } W_{\vec{F}}?$$

b. Compute the exterior derivative of $x^2\, dx + y^2 z\, dy + xy\, dz$ using Theorem 6.7.4. Show that it is the same as $\Phi_{\vec{\nabla}\times\vec{F}}$.

6.21 a. There is an exponent m such that

$$\vec{\nabla} \cdot (x^2 + y^2 + z^2)^m \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0; \quad \text{find it.}$$

Exercise 6.21, part b: The subscript on Φ may be hard to read. It is $r^{2m}\vec{r}$.

*b. More generally, there is an exponent m (depending on n) such that the $(n-1)$-form $\Phi_{r^{2m}\vec{r}}$ has exterior derivative 0, when $\vec{r}$ is the vector field $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, and $r = |\vec{r}|$. Can you find it? (Start with $n = 1$ and $n = 2$.)

6.22 a. Find the unique polynomial p such that $p(1) = 1$ and such that if

$$\omega = x\, dy \wedge dz - 2zp(y)\, dx \wedge dy + yp(y)\, dz \wedge dx,$$

then $\mathbf{d}\omega = dx \wedge dy \wedge dz$.

b. For this polynomial p, find the integral $\int_S \omega$, where S is that part of the sphere $x^2+y^2+z^2 = 1$ where $z \geq \sqrt{2}/2$, oriented by the outward-pointing normal.

6.23 a. Compute the exterior derivative of the 2-form

$$\varphi = \frac{1}{(x^2+y^2+z^2)^{3/2}} \Big(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \Big).$$

b. Compute the integral of φ over the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the outward-pointing normal.

c. Compute the integral of φ over the boundary of the cube of side 4, centered at the origin, and oriented by the outward-pointing normal.

d. Can φ be written $\mathbf{d}\psi$ for some 1-form ψ on $\mathbb{R}^3 - \{\mathbf{0}\}$?

6.24 Let S be the surface of equation $z = 9 - y^2$, oriented by the upward-pointing normal.

a. Sketch the piece $X \subset S$ where $x \geq 0$, $z \geq 0$, and $y \geq x$, indicating carefully the boundary orientation.

b. Give a parametrization of X, being careful about the domain of the parametrizing map and whether it is orientation preserving.

c. Find the work of the vector field $\begin{bmatrix} 0 \\ xz \\ 0 \end{bmatrix}$ around the boundary of X.

6.25 Let $U \subset \mathbb{R}^3$ be a subset bounded by a surface S, to which we will give the boundary orientation. What relation is there between the volume of U and the flux $\int_S \Phi_{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$?

6.26 Let $\vec{F}$ be the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \\ 0 \end{bmatrix}$, where F_1 and F_2 are defined on all of $\mathbb{R}^2$. Suppose $D_2F_1 = D_1F_2$. Show that there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \vec{\nabla} f$.

6.27 Show that the electromagnetic field of a charge q moving in the direction of the x-axis at constant speed v is

$$\vec{\mathbf{E}} = \frac{q\gamma}{4\pi\Big((\gamma x - \gamma vt)^2 + y^2 + z^2\Big)^{3/2}} \begin{bmatrix} x - vt \\ y \\ z \end{bmatrix}$$

$$\vec{\mathbf{B}} = \frac{v}{c}\; \frac{q\gamma}{4\pi\Big((\gamma x - \gamma vt)^2 + y^2 + z^2\Big)^{3/2}} \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix}, \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Exercise 6.27: Equation 6.12.75 relates the coordinates x, y, z, t of a frame of reference moving at speed v in the direction of the x-axis with respect to a frame with coordinates x', y', z', t'. Equation 6.12.24 gives the electromagnetic field of a charge at rest.

It is often said that the magnetic field is a relativistic side effect of the electric field of moving charges. Exercise 6.27 illustrates this: with the particle at rest the magnetic field is $\vec{\mathbf{0}}$, but a moving charge will deflect a magnetic needle. Note the v/c in the magnetic field of the moving charge. At ordinary (human) speeds, the magnetic field will be extremely small.

6.28 Find a 1-form φ such that $\mathbf{d}\varphi = y\, dz \wedge dx - x\, dy \wedge dz$.

6.29 Let $U \subset \mathbb{R}^3$ be a 3-dimensional piece-with-boundary.

a. What does Stokes's theorem say about $\int_U \mathbf{d}\mathbb{F}$?

b. What does Stokes's theorem say about $\int_U \mathbf{d}\mathbb{M}$?

6.30 Let $S \subset \mathbb{R}^3$ be a smooth oriented surface, and $X \subset S$ a 2-dimensional piece-with-boundary. Let $I = [t_0, t_1]$ be a time interval.

a. Show that $V = S \times I$ is a 3-dimensional piece-with-boundary of $\mathbb{R}^4$.

b. What does Stokes's theorem say about $\int_V \mathbf{d}\mathbb{F}$? Show that if we divide the integral by $t_1 - t_0$ and let t_1 tend to t_0, we find Faraday's law.

c. What does Stokes's theorem say about $\int_V \mathbf{d}\mathbb{M}$? Show that if we divide the integral by $t_1 - t_0$ and let t_1 tend to t_0, we find Ampère's law.

Exercise 6.30: The surface S might be very complicated, for instance, the boundary of a cloud. Understanding the charge on the boundary of a cloud is essential for understanding lightning and thunderstorms.

6.31 Let $\vec{F}_n = \frac{\vec{\mathbf{x}}}{|\vec{\mathbf{x}}|^n}$ be the vector field defined in Example 6.7.7, and let S_R^{n-1} be the sphere $|\vec{\mathbf{x}}| = R$, oriented by the outward-pointing normal. Show that $\int_{S_R^{n-1}} \Phi_{\vec{F}_n}$ does not depend on R, and that it equals $\text{vol}_{n-1}(S_1^{n-1})$.

6.32 Compute the integral $\int_S \Phi_{\vec{F}}$, where $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -x^2yz \\ y \\ (z^2-1)xy \end{bmatrix}$, and S is the part of the parabolic cylinder of equation $y = 9 - x^2$ where $y \geq 0$ and $0 \leq z \leq 1$, oriented by the transverse vector field $\vec{\mathbf{e}}_2$.

6.33 Let V, W, be two finite-dimensional real vector spaces, oriented by

$$\Omega^V : \mathcal{B}(V) \to \{\pm 1\} \quad \text{and} \quad \Omega^W : \mathcal{B}(W) \to \{\pm 1\}.$$

Show that

$$\Omega^{V\times W}(\{v\},\{w\}) \stackrel{\text{def}}{=} \Omega^V(\{v\})\,\Omega^W(\{w\})$$

orients $V \times W$.

6.34 Let $\vec{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $r = |\vec{\mathbf{r}}| = \sqrt{x^2+y^2+z^2}$. Show that

$$\mathbb{F} = qW_{\vec{\mathbf{r}}/r^3} \wedge c\,dt$$

is an electromagnetic field. What are the corresponding charges and currents?

Exercise 6.34 asks you to show that $\mathbf{d}\mathbb{F} = 0$ and to compute $\mathbf{d}\mathbb{M}$. In both cases, these exterior derivatives are "in the sense of distributions"; see Remark 6.12.1 (page 676).

Remarks. 1. Exercise 6.34 isn't a straightforward computation. For $\mathbb{F}$ to be an electromagnetic field, we must have $\mathbf{d}\mathbb{F} = 0$ everywhere, and it isn't clear what this means when $r = 0$. In Example 6.13.1 we saw that trouble may be hiding at points where a form is not defined. We deal with this as follows.

A 3-dimensional piece-with-boundary $X \subset \mathbb{R}^4$ will be called r*-adapted* if locally near points where $r = 0$ it represents t as a function of x, y, and z; in that case we write $X_\epsilon = X - \{r < \epsilon\}$ and we define

$$\partial_{inn} X_\epsilon \subset \partial X_\epsilon$$

to be the subset where $r = \epsilon$, with the boundary orientation. If φ is a 2-form on $\mathbb{R}^4 - \{r = 0\}$ and $X \subset \mathbb{R}^4$ is an r-adapted oriented 3-dimensional piece-with-boundary, we define

$$\int_X \mathbf{d}\varphi \overset{\text{def}}{=} \lim_{\epsilon \to 0} \left(\int_{X_\epsilon} \mathbf{d}\varphi - \int_{\partial_{inn} X_\epsilon} \varphi \right).$$

If φ is well defined where $r = 0$, then, by Stokes's theorem,

$$\int_X \mathbf{d}\varphi = \int_{X_\epsilon} \mathbf{d}\varphi + \int_{X - X_\epsilon} \mathbf{d}\varphi = \int_{X_\epsilon} \mathbf{d}\varphi - \int_{\partial_{inn} X_\epsilon} \varphi,$$

so the formula is correct in that case, and otherwise the boundary term captures whatever is hiding on $\{r = 0\}$.

2. Note that

$$\mathbf{d}\left(\frac{1}{r}\right) = -W_{\vec{\mathbf{r}}/r^3}.$$

This contrasts with Example 6.13.1. There we had

$$\mathbf{d} \arctan \frac{y}{x} = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

But $\arctan \frac{y}{x}$ *is not* a well-defined function on $\mathbb{R}^2 - \{\mathbf{0}\}$, whereas $1/r$ is a well-defined function on $\mathbb{R}^3 - \{\mathbf{0}\}$.

Forms were defined to be integrands, so it is quite reasonable to define $\mathbf{d}\mathbb{F}$ via its integrals over oriented pieces-with-boundary. It is also quite reasonable to restrict to r-adapted pieces: pieces-with-boundary for which, locally near points where $r = 0$, the piece X is the graph of a map expressing t as a function of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Pieces that are not r-adapted intersect the locus $r = 0$ in exceptional ways, and it may be difficult to say what share of whatever nastiness is hiding there is carried by X. Such pieces are exceptional; by budging them arbitrarily little we can make them r-adapted. △