6.13 Review exercises for chapter 6

6.1 Which of the following are numbers? Identify those that are not. (Below, B is an $n \times n$ matrix and φ and ψ are both 1-forms on $\mathbb{R}^3$; $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are vectors in $\mathbb{R}^3$; f is integrable.)

a. $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$	b. $dx_1 \wedge dx_2(\vec{\mathbf{v}}, \vec{\mathbf{w}})$	c. $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$
d. $\det B$	e. $\operatorname{rank} B$	f. $\operatorname{tr} B$
g. $\dim A^k(\mathbb{R}^n)$	h. $\lvert\vec{\mathbf{v}}\rvert$	i. $A^k(\mathbb{R}^k)$
j. $\varphi \wedge \psi(\vec{\mathbf{v}}, \vec{\mathbf{w}})$	k. $\int_{\mathbb{R}} f(x)\, dx$	l. $\operatorname{sgn}(\sigma)$

6.2 Let $\vec{\mathbf{F}}$ be a vector field in $\mathbb{R}^n$, let $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}$ be vectors in $\mathbb{R}^n$ and let φ be the $(n-1)$-form on $\mathbb{R}^n$ given by

$$\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}) = \det[\vec{\mathbf{F}}(\mathbf{x}), \vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}].$$

Write φ as a linear combination of elementary $(n-1)$-forms on $\mathbb{R}^n$, in terms of the coordinates of $\vec{\mathbf{F}}$.

6.3 Use definition 6.1.19 to write the following wedge product as a combination of values of φ and ψ evaluated on appropriate vectors (as in equations 6.1.35 and 6.1.40):

$\varphi \wedge \psi(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4)$, where φ is a 1-form and ψ is a 3-form.

6.4 Set up each of the following integrals of form fields over parametrized domains as an ordinary multiple integral.

a. $\int_{[\gamma(I)]} y^2 dy + x^2 dz$, where $I = [0, a]$ and $\gamma(t) = \begin{pmatrix} t^3 \\ t^2+1 \\ t^2-1 \end{pmatrix}$

b. $\int_{[\gamma(U)]} \sin y^2\, dx \wedge dz$, where $U = [0, a] \times [0, b]$, and $\gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^2 - v \\ uv \\ v^4 \end{pmatrix}$

6.5 Find a 1-form field on $\mathbb{R}^2$ whose restriction to the circle of radius 1 centered at $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ orients that circle in the clockwise direction.

6.6 Find a 2-form that orients the surface $S \subset \mathbb{R}^3$ given by $x^2 + y^3 + z^4 = 1$.

6.7 Consider the manifold $S^3 \subset \mathbb{R}^4$ of equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

a. Show that $dx_1 \wedge dx_2 \wedge dx_3$ is not an orientation of S^3.

b. Show that $\omega_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3) = \det[\mathbf{x}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$ is an orientation.

6.8 a. Show that the locus $M \subset \mathbb{R}^4$ given by $x_1^2 + x_2^2 + x_1 x_4^2 = 1$ is a smooth manifold.

b. Find a 3-form field to orient M.

6.9 In example 6.4.3 we saw that

$$\gamma_1(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \gamma_2(\theta) = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

give opposite orientations. Confirm (proposition 6.4.8) that

$$\det[\mathbf{D}(\gamma_2^{-1} \circ \gamma_1)] < 0.$$

6.10 Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be coordinates in $\mathbb{C}^2$. Compute the integral of $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ over the part of the locus of equation $z_2 = z_1^k$ where $|z_1| < 1$, oriented by $dx_1 \wedge dy_1$.

6.11 For the following 1-forms, write down the corresponding vector field. Sketch the vector field. Describe a path over which the work of the 1-form would be small. Describe a path over which the work would be large.

a. $(x^2 + y^2)\,dz$, on $\mathbb{R}^3$ b. $y\,dx - x\,dy - z\,dz$, on $\mathbb{R}^3$

6.12 a. In $\mathbb{R}^2$, a vector field defines two 1-forms, the work and the flux. Show that they are related by formula

$$W_{\vec{F}}(\vec{\mathbf{v}}) = \Phi_{\vec{F}}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{\mathbf{v}}\right).$$

b. What does the transformation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ correspond to geometrically?

c. Can you explain why the work and the flux on $\mathbb{R}^2$ are related by the formula in part a?

6.13 Find the flux of the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y \\ -z \\ yz \end{bmatrix}$, through S, where S is the part of the cone $z = \sqrt{x^2 + y^2}$ where $x, y \geq 0$, $x^2 + y^2 \leq R$, and it is oriented by the inward-pointing normal (i.e., the flux measures the amount flowing into the cone).

6.14 Let S be the part of the surface of equation $z = \sin xy + 2$ where $x^2 + y^2 \leq 1$ and $x \geq 0$, oriented by the upward-pointing normal. What is the flux of the vector field $\begin{bmatrix} 0 \\ 0 \\ x + y \end{bmatrix}$ through S?

6.15 Consider the manifold $S^3 \subset \mathbb{R}^4$ of equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, oriented by the form field ω defined by $\omega_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3) = \det[\mathbf{x}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$. Let X be the subset of M where $x_4 \leq 0$.

a. Show that X is a piece-with-boundary of S^3.

b. Find a basis for the tangent space $T_{\mathbf{x}}(\partial X)$ at the point $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \partial X$ which is direct for the boundary orientation associated to the orientation ω of S^3.

6.16 a. Compute from the definition the derivative of $xy\ dz$.

b. Compute the same derivative using the formulas given in theorem 6.7.3, stating clearly at each stage what property you are using.

6.17 a. Let $\varphi = xyz\,dy$. Compute from the definition the number

$$d\varphi\left(P_{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}(\vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3)\right).$$

b. What is $d\varphi$? Use your result to check the computation in part a.

6.18 Let $\vec{\mathbf{r}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be the radial vector field in $\mathbb{R}^n$.

Exercise 6.18 gives another way to derive equation 5.3.54.

a. Show that $d(\Phi_{\vec{\mathbf{r}}}) = n(dx_1 \wedge \cdots \wedge dx_n)$.

b. Let $B_1^n(\mathbf{0})$ and S^{n-1} be the unit ball and the unit sphere in $\mathbb{R}^n$, the ball with the standard orientation and the sphere with the boundary orientation. Use Stokes's theorem to prove

$$\mathrm{vol}_n\Big(B_1^n(\mathbf{0})\Big) = \frac{1}{n}\,\mathrm{vol}_{n-1}(S^{n-1}).$$

6.19 Using the formulas of theorem 6.8.3, prove the equations

$$\mathrm{curl}\,(\mathrm{grad}\, f) = \mathbf{0} \quad \text{and} \quad \mathrm{div}(\mathrm{curl}\, \vec{F}) = 0$$

for any function f and any vector field $\vec{F}$ (at least of class C^2).

6.20 a. For what vector field $\vec{F}$ is the 1-form on $\mathbb{R}^3$

$$x^2\, dx + y^2 z\, dy + xy\, dz \quad \text{the work form field } W_{\vec{F}}?$$

b. Compute the exterior derivative of $x^2\, dx + y^2 z\, dy + xy\, dz$ using theorem 6.7.3. Show that it is the same as $\Phi_{\vec{\nabla}\times\vec{F}}$.

6.21 a. There is an exponent m such that

$$\vec{\nabla} \cdot (x^2 + y^2 + z^2)^m \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0; \quad \text{find it.}$$

Exercise 6.21, part b: The subscript on Φ may be hard to read. It is $r^{2m}\vec{r}$.

*b. More generally, there is an exponent m (depending on n) such that the $(n-1)$-form $\Phi_{r^{2m}\vec{r}}$ has exterior derivative 0, when $\vec{r}$ is the vector field $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, and $r = |\vec{r}|$. Can you find it? (Start with $n = 1$ and $n = 2$.)

6.22 a. Find the unique polynomial p such that $p(1) = 1$ and such that if

$$\omega = x\, dy \wedge dz - 2zp(y)\, dx \wedge dy + yp(y)\, dz \wedge dx,$$

then $d\omega = dx \wedge dy \wedge dz$.

b. For this polynomial p, find the integral $\int_S \omega$, where S is that part of the sphere $x^2+y^2+z^2 = 1$ where $z \geq \sqrt{2}/2$, oriented by the outward-pointing normal.

6.23 a. Compute the exterior derivative of the 2-form

$$\varphi = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

b. Compute the integral of φ over the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the outward-pointing normal.

c. Compute the integral of φ over the boundary of the cube of side 4, centered at the origin, and oriented by the outward-pointing normal.

d. Can φ be written $d\psi$ for some 1-form ψ on $\mathbb{R}^3 - \{\mathbf{0}\}$?

6.24 Let S be the surface of equation $z = 9 - y^2$, oriented by the upward-pointing normal.

a. Sketch the piece $X \subset S$ where $x \geq 0$, $z \geq 0$ and $y \geq x$, indicating carefully the boundary orientation.

b. Give a parametrization of X, being careful about the domain of the parametrizing map and whether it is orientation preserving.

c. Find the work of the vector field $\begin{bmatrix} 0 \\ xz \\ 0 \end{bmatrix}$ around the boundary of X.

6.25 Let $U \subset \mathbb{R}^3$ be a subset bounded by a surface S, which we will give the boundary orientation. What relation is there between the volume of U and the flux $\int_S \Phi_{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$?

6.26 Compute the integral $\displaystyle\int_S \Phi_{\vec{F}}$, where $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -x^2yz \\ y \\ (z^2-1)xy \end{bmatrix}$ and S is the part of the parabolic cylinder of equation $y = 9 - x^2$ where $y \geq 0$ and $0 \leq z \leq 1$, oriented by $\vec{\mathbf{e}}_2$.

6.27 Let $\vec{F}$ be the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \\ 0 \end{bmatrix}$.

Suppose $D_2F_1 = D_1F_2$. Show that there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \vec{\nabla} f$.

6.28 Find a 1-form φ such that $d\varphi = y\,dz \wedge dx - x\,dy \wedge dz$.