# 4.12 Review exercises for chapter 4

**4.1** Show that if $C \in \mathcal{D}(\mathbb{R}^m)$, then $\mathbf{1}_C$ is integrable.

Exercise 4.1:

$U_N(\mathbf{1}_C) = L_N(\mathbf{1}_C)$.

**4.2** An integrand should take a piece of the domain, and return a number, in such a way that if we decompose a domain into little pieces, evaluate the integrand on the pieces and add, the sums should have a limit as the decomposition becomes infinitely fine (and the limit should not depend on how the domain is decomposed). What will happen if we break up $[0,1]^2$ into rectangles defined by $a < x < b$ and $c < y < d$ and assign one of the numbers below to each rectangle?

a. $|ac - bd|$ b. $(ad - bc)^2$.

**4.3** Let $A$ be an $n \times n$ matrix of integers, viewed as a map $\mathbb{Z}^n \to \mathbb{Z}^n$. Which of the following are true?

1. $\ker A = 0 \implies A$ is onto.
2. $A$ onto $\implies \ker A = 0$.
3. $\det A \neq 0 \implies \ker A = 0$.
4. $\det A \neq 0 \implies A$ is onto.

**4.4** Which elementary matrices are permutation matrices? Describe the corresponding permutations.

**4.5** Evaluate $\displaystyle\lim_{N\to\infty} \frac{1}{N^2} \sum_{k=1}^{N} \sum_{l=1}^{2N} e^{\frac{k+l}{N}}$.

**4.6** What are the expectation, variance, and standard deviation of the random variable $f(x) = x$, for the following probability densities.

a. $\mu(x) = e^{-x}\mathbf{1}_{[0,\infty]}$ b. $\mu(x) = \dfrac{1}{x^2}\mathbf{1}_{[0,\infty]}$

**4.7** Let $A$ and $B$ be two disjoint bodies, with densities $\mu_1$ and $\mu_2$ and masses $M(A)$ and $M(B)$. Set $C = A \cup B$. Show that the center of gravity of $C$ is

$$\overline{\mathbf{x}}(C) = \frac{M(A)\overline{\mathbf{x}}(A) + M(B)\overline{\mathbf{x}}(B)}{M(A) + M(B)}.$$

**4.8** Prove corollary 4.3.11.

**4.9** Let $X$ be a subset of $\mathbb{R}^n$ such that for any $\epsilon > 0$, there exists a sequence of pavable sets $B_i,\ i = 1, 2, \ldots$ satisfying $X \subset \bigcup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} \mathrm{vol}_n(B_i) < \epsilon$. Show that $X$ has measure 0.

**4.10** Give an explicit upper bound (in terms of $N$) for the number of cubes in $\mathcal{D}_N(\mathbb{R}^3)$ needed to cover the unit sphere $S^2 \subset \mathbb{R}^3$, such that the volume of the cubes tends to 0 as $N$ tends to infinity.

**4.11** Write each of the following double integrals as iterated integrals in two ways, and compute them:

a. The integral of $\sin(x + y)$ over the region $x^2 < y < 2$.
b. The integral of $x^2 + y^2$ over the region $1 \le |x|, |y| \le 2$.

**4.12** Compute the integral of the function $z$ over the region $R$ described by the inequalities $x > 0,\ y > 0,\ z > 0,\ x + 2y + 3z < 1$.

**4.13** a. If $f\begin{pmatrix} x \\ y \end{pmatrix} = a + bx + cy$, what are

$$\int_0^1 \int_0^2 f\begin{pmatrix} x \\ y \end{pmatrix} |dx\, dy| \quad \text{and} \quad \int_0^1 \int_0^2 \left( f\begin{pmatrix} x \\ y \end{pmatrix} \right)^2 |dx\, dy|?$$

b. Let $f$ be as in part a. What is the minimum of $\int_0^1 \int_0^2 \left( f\begin{pmatrix} x \\ y \end{pmatrix} \right)^2 |dx\, dy|$ among all functions $f$ such that

$$\int_0^1 \int_0^2 f\begin{pmatrix} x \\ y \end{pmatrix} |dx\, dy| = 1?$$

**4.14** What is the $z$-coordinate of the center of gravity of the region

$$\frac{x^2}{(z^3-1)^2} + \frac{y^2}{(z^3+1)^2} \leq 1, \quad 0 \leq z \leq 1\,?$$

**4.15** Show that there exist $c$ and $u$ such that when $f$ is any polynomial of degree $d \leq 3$,

$$\int_{-1}^1 f(x) \frac{1}{\sqrt{1-x^2}}\, dx = c\Big(f(u) + f(-u)\Big).$$

**4.16** Repeat exercise 4.6.4 a–d, but this time for the weight $e^{-x^2}$ and limits of integration $-\infty$ to $\infty$; i.e., find points $x_i$ and $w_i$ such that

$$\int_{-\infty}^{\infty} p(x) e^{-x^2}\, dx = \sum_{i=0}^{k} w_i p(x_i)$$

is true for all polynomials of degree $\leq 2k-1$.

e. For each of the five values of $m$ in part d, approximate

$$\int_{-\infty}^{\infty} e^{-x^2} \cos x\, dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2}\, dx.$$

Compare the approximations with the exact values.

**4.17** Check part c of theorem 4.8.14 when $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; that is, show that $[\mathbf{D}\det(A)]B = \det A\ \text{tr}(A^{-1}B)$.

Exercise 4.18: Start with corollary 4.8.16, and set

$$C = P,\ D = AP^{-1}.$$

This proves the formula when $C$ is invertible. Complete the proof by showing that if $C_n$ is a sequence of matrices converging to $C$, and $\text{tr}(C_n D) = \text{tr}(DC_n)$ for all $n$, then $\text{tr}(CD) = \text{tr}(DC)$.

**4.18** Show that if $A$ and $B$ are $n \times n$ matrices, then $\text{tr}(AB) = \text{tr}(BA)$.

**4.19** What is the $n$-dimensional volume of the region

$$\{\, \mathbf{x} \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i = 1, \dots, n \text{ and } x_1 + 2x_2 + \cdots + nx_n \leq n \,\}?$$

**4.20** a. Find an expression for the area of the parallelogram spanned by $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, in terms of $|\vec{\mathbf{v}}_1|, |\vec{\mathbf{v}}_2|$, and $|\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2|$.

b. Prove Heron's formula: A triangle with sides of length $a, b, c$, has area

$$\sqrt{p\,(p-a)(p-b)(p-c)}\ , \quad \text{where} \quad p = \frac{a+b+c}{2}.$$

**4.21** a. Sketch the curve in the plane given in polar coordinates by the equation $r = 1 + \sin\theta, \quad 0 \leq \theta \leq 2\pi$.

b. Find the area that the curve encloses.

**4.22** A semicircle of radius $R$ has density $\rho\begin{pmatrix} x \\ y \end{pmatrix} = m(x^2+y^2)$ proportional to the square of the distance to the center. What is its mass?

**4.23** a. Let $Q$ be the part of the unit ball $x^2+y^2+z^2 \le 1$ where $x, y, z \ge 0$. Using spherical coordinates, set up $\int_Q (x+y+z)\,|d^3\mathbf{x}|$ as an iterated integral.

b. Compute the integral.

**4.24** Let $\mathbf{x} \in \mathbb{R}^n$. For what values of $p \in \mathbb{R}$ does $\int_{B_1(\mathbf{0})} |\mathbf{x}|^p\,|d^n\mathbf{x}|$ exist as a Lebesgue integral? (The answer depends on $n$.)

**4.25** Let $A$ be the region defined the inequalities $x^2+y^2 \le z \le 1$. What is the center of gravity of $A$?

**4.26** In this exercise we will show that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. This function is not Lebesgue integrable, and the integral should be understood as

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{a\to\infty}\int_0^a \frac{\sin x}{x}\,dx.$$

a. Show that for all $0 < a < b < \infty$,

$$\int_a^b \left(\int_0^\infty e^{-px}\sin x\,dx\right)dp = \int_0^\infty \left(\int_a^b e^{-px}\sin x\,dp\right)dx.$$

b. Use part a to show

$$\arctan b - \arctan a = \int_0^\infty \frac{(e^{-ax}-e^{-bx})\sin x}{x}\,dx.$$

c. Why does theorem 4.11.4 not imply that

$$\lim_{a\to 0}\lim_{b\to\infty}\int_0^\infty \frac{(e^{-ax}-e^{-bx})\sin x}{x}dx = \int_0^\infty \frac{\sin x}{x}\,dx? \qquad (1)$$

Part d: Remember that the next omitted term is a bound for the error for each partial sum.

d. Prove that equation (1) is true anyway. The following lemma is the key: If $c_n(t) > 0$ are monotone increasing functions of $t$, with $\lim_{t\to\infty} c_n(t) = C_n$, and decreasing as a function of $n$ for each fixed $t$, tending to 0, then

$$\lim_{t\to\infty}\sum_{n=1}^\infty (-1)^n c_n(t) = \sum_{n=1}^\infty (-1)^n C_n.$$

e. Write

$$\int_0^\infty \frac{(e^{-ax}-e^{-bx})\sin x}{x}\,dx = \sum_{n=0}^\infty \int_{k\pi}^{(k+1)\pi} (-1)^k\frac{(e^{-ax}-e^{-bx})|\sin x|}{x}\,dx,$$

and use part d to prove the equation $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

**4.27** Let $a_1, a_2, \dots$ be a list of the rationals in $[0,1]$. Consider

$$f(x) = \sum_{k=1}^\infty \frac{1}{2^k}\frac{1}{\sqrt{|x-a_k|}}.$$

a. Show that $f$ is L-integrable on $[0,1]$.

b. Show that the series converges for all $x$ except $x$ on a set of measure 0.

*c. Find an $x$ for which the series converges. (This depends on the order chosen.)

**4.28** What are the expectation, variance, and standard deviation of the function (random variable) $f(x) = x$ for the probability densitiy

$$\mu(x) = \frac{1}{x^2}\mathbf{1}_{[0,\infty]},$$

where the sample space is all of $\mathbb{R}$?

**4.29** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(u) = au + b\bar{u}$, where we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the standard way. Show that

$$\det T = |a|^2 - |b|^2 \quad \text{and} \quad \|T\| = |a| + |b|.$$