## 0.6 Infinite sets

One reason set theory is accorded so much importance is that Georg Cantor (1845–1918) discovered that two infinite sets need not have the same "number" of elements; there isn't just one infinity. You might think this is just obvious; for example, that clearly there are more whole numbers than even whole numbers. But with the definition Cantor gave, two sets $A$ and $B$ have the same number of elements (the same *cardinality*) if you can set up a bijective correspondence between them (i.e., a mapping that is one to one and onto). For instance,

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & \dots \end{array} \qquad 0.6.1$$

establishes a bijective correspondence between the natural numbers and the even natural numbers. Similarly,

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \dots \end{array}$$

establishes a bijective correspondence between $\mathbb{N}$ and $\mathbb{Z}$. More generally, any set whose elements you can list has the same cardinality as $\mathbb{N}$. But in 1873 Cantor discovered that $\mathbb{R}$ does not have the same cardinality as $\mathbb{N}$: it has a *bigger infinity of elements*. Indeed, imagine making any infinite list of real numbers, say between 0 and 1, so that written as decimals, your list might look like

$$\begin{array}{l} .\mathbf{1}5436278645342982376349065236734754875 7\dots \\ .9\mathbf{8}7354621943756598673562940657349327658\dots \\ .22\mathbf{9}57352190356435542303546552339008074 2\dots \\ .104\mathbf{7}5201874626765320936572368907656578 7\dots \\ .0263\mathbf{2}8560082356835654432879897652377327\dots \\ \dots . \end{array} \qquad 0.6.2$$

Figure 0.6.1.
Georg Cantor (1845–1918)
After thousands of years of philosophical speculation about the infinite, Cantor found a fundamental notion that had been completely overlooked.

Recall (section 0.3) that $\mathbb{N}$ is the "natural numbers" $0, 1, 2, \dots$; $\mathbb{Z}$ is the integers; $\mathbb{R}$ is the real numbers.

It would seem likely that $\mathbb{R}$ and $\mathbb{R}^2$ have different infinities of elements, but that is not the case (see exercise A1.5).

Figure 0.6.2.
Joseph Liouville (1809–1882)

Now consider the decimal $.18972\dots$ formed by the elements of the diagonal digits (in bold in equation 0.6.2), and modify it (almost any way you want) so that every digit is changed, for instance according to the rule "change 7s to 5s and change anything that is not a 7 to a 7": in this case, your number becomes $.77757\dots$. Clearly this last number does not appear in your list: it is not the $n$th element of the list, because it doesn't have the same $n$th decimal.

Sets that can be put in one-to-one correspondence with the natural numbers are called *countable*. Those that cannot are called *uncountable*; the set $\mathbb{R}$ of real numbers is uncountable.

### Existence of transcendental numbers

An *algebraic number* is a root of a polynomial equation with integer coefficients: the rational number $p/q$ is algebraic, since it is a solution of

$qx - p = 0$, and so is $\sqrt{2}$, since it is a root of $x^2 - 2 = 0$. A number that is not algebraic is called *transcendental*. In 1851 Joseph Liouville came up with the transcendental number (now called the *Liouvillian number*)

$$\sum_{n=1}^{\infty} \frac{1}{10^{n!}} = 0.11000100000000000000000100\ldots, \qquad 0.6.3$$

the number with 1 in every position corresponding to $n!$ and 0s elsewhere. In 1873 Charles Hermite proved a much harder result, that $e$ is transcendental. But Cantor's work on cardinality made it obvious that there must exist uncountably many transcendental numbers: all those real numbers left over when one tries to put the real numbers in one-to-one correspondence with the algebraic numbers.

FIGURE 0.6.3.
Charles Hermite (1822–1901)
For Hermite, there was something scandalous about Cantor's proof of the existence of infinitely many transcendental numbers, which required no computations and virtually no effort and failed to come up with a single example. "This isn't math," he exclaimed, "it's theology."

Here is one way to show that the algebraic numbers are countable. First list the polynomials $a_1x+a_0$ of degree $\le 1$ with integer coefficients satisfying $|a_i| \le 1$, then the polynomials $a_2x^2 + a_1x + a_0$ of degree $\le 2$ with $|a_i| \le 2$, etc. The list starts

$$\begin{aligned} &-x-1, -x+0, -x+1, -1, 0, 1, x-1, x, x+1, -2x^2-2x-2, \\ &\quad -2x^2-2x-1, -2x^2-2x, -2x^2-2x+1, -2x^2-2x+2, \ldots. \end{aligned} \qquad 0.6.4$$

(The polynomial $-1$ in equation 0.6.4 is $0 \cdot x - 1$.) Then we go over the list, crossing out repetitions.

Next we write a second list, putting first the roots of the first polynomial in equation 0.6.4, then the roots of the second polynomial, etc.; again, go through the list and cross out repetitions. This lists all algebraic numbers, showing that they form a countable set.

## Other consequences of different cardinalities

Two sets $A$ and $B$ have the same cardinality (denoted $A \asymp B$) if there exists an invertible mapping $A \to B$. A set $A$ is countable if $A \asymp \mathbb{N}$, and it has *the cardinality of the continuum* if $A \asymp \mathbb{R}$. We will say that the cardinality of a set $A$ is at most that of $B$ (denoted $A \preceq B$) if there exists a one-to-one map from $A$ to $B$. Bernstein's theorem, sketched in exercise 0.6.5, shows that if $A \preceq B$ and $B \preceq A$, then $A \asymp B$.

The fact that $\mathbb{R}$ and $\mathbb{N}$ have different cardinalities raises all sorts of questions. Are there other infinities besides those of $\mathbb{N}$ and $\mathbb{R}$? We will see in proposition 0.6.1 that there are infinitely many.

Let $\mathcal{P}(E)$ denote the set of all subsets of $E$, called the *power set* of $E$. Clearly for any set $E$ there exists a one-to-one map $f: E \to \mathcal{P}(E)$; for instance, the map $f(a) = \{a\}$. So the cardinality of $E$ is at most that of $\mathcal{P}(E)$. In fact, it is strictly less. If $E$ is finite and has $n$ elements, then $\mathcal{P}(E)$ has $2^n$ elements, clearly more than $E$ (see exercise 0.6.2). Proposition 0.6.1 says that this is still true if $E$ is infinite.

**Proposition 0.6.1.** *A mapping $f : E \to \mathcal{P}(E)$ is never onto.*