Introduction

This volume is the third of four volumes devoted to Teichmüller theory and its applications to geometry, topology, and dynamics. Volume 1 gave an introduction to Teichmüller theory. Volumes 2 through 4 prove four theorems by William Thurston:

- ◇ The classification of homeomorphisms of surfaces
- ◇ The topological characterization of rational maps
- ◇ The hyperbolization theorem for 3-manifolds that fiber over the circle
- ◇ The hyperbolization theorem for Haken 3-manifolds

These theorems have been quite difficult to approach, in part because Thurston never published complete proofs of any of them. Volume 2, *Surface Homeomorphisms and Rational Functions*, proved the first two theorems. The present volume is devoted to the third.

There is an elephant in the room: Perelman's proof [68, 69] of Thurston's "eight geometries" conjecture, based on Hamilton's study of the Ricci flow, [34, 35]. In some sense, it makes the main theorems of Volume 3 and 4 obsolete. On the other hand, Perelman's proof describes very well the possible singularities that manifolds may acquire under Ricci flow, but gives little insight about the geometry of the complements of these singularities. Thurston's proofs are filled with detailed information about this geometry.

In Thurston's approach [85, 86, 87], the results, which sound more or less unrelated, are linked by a common thread: each one goes from topology to geometry via complex analysis. Each says that either a topological problem has a natural geometry, or there is an understandable obstruction.

The techniques used to prove these results were so novel that they have barely entered the accepted methods of mathematicians in the field. In part this is due to the discomfort many topologists and geometers feel when faced with complex analysis and Riemann surfaces: they much prefer to define Teichmüller space as the space of marked hyperbolic surfaces. But Thurston was a great topologist, a great geometer, and a great complex analyst. The key constructs of the theorems above – the pull-back map, the skinning map, and the Bers simultaneous uniformization theorem – require topology and geometry, but they also require complex analysis: they cannot be understood in terms of hyperbolic structures.

The proofs are closely related: you use the topology to set up an analytic map σ from a Teichmüller space to itself. The geometry arises from a fixed point of σ. If there is no fixed point, then a sequence obtained by iterating σ diverges, and this divergence gives rise to the obstruction.

Proving the third theorem, which we do in Chapter 13, requires quite a bit of preliminary material.

Hyperbolic geometry in dimension 2 is covered fairly seriously in Chapters 2 and 3 of Volume 1. Hyperbolic geometry in higher dimensions is much richer and more difficult. We focus on three dimensions, because in three dimensions complex analysis is a powerful tool: one can view the "boundary at infinity" of hyperbolic space as the Riemann sphere.

In Chapter 11 we cover some topics that will be essential for us. Beyond the basics, topics covered include Jorgensen's inequality, the Margulis lemma, algebraic and geometric convergence and the Chabauty topology, the Klein-Maskit combination theorems, fundamental domains and the geometrization of the complement of the figure-eight knot, and several properties of geometrically finite Kleinian groups.

Chapter 12 begins with three great theorems due to Ahlfors, McMullen, and Mostow. These theorems are concerned with the construction of Beltrami forms on $\partial\overline{\mathbb{H}}^3$ that are invariant under a Kleinian group Γ. Since they show that such Beltrami forms are very restricted, they are called *rigidity theorems*. We then show that the central hypothesis of McMullen's rigidity theorem holds for quasi-Fuchsian groups – a result we will need to prove the hyperbolization theorem for 3-manifolds that fiber over the circle in Chapter 13. In doing so we introduce several topics, including laminations and pleated surfaces, of great interest in their own right.

Finding a hyperbolic structure on a 3-manifold is equivalent to finding a Kleinian group isomorphic to its fundamental group. In Chapter 13 we do this for 3-manifolds that fiber over the circle, exhibiting the Kleinian group as an appropriate limit. The Bers compactness theorem in Section 13.4 is a very beautiful result asserting that some limits of Kleinian groups exist, unfortunately not quite the ones we need.

Finding the limits we need requires yet more machinery, including $\mathbb{R}$-trees and Γ-trees, leaf spaces of measured foliations, Hatcher's construction relating measured foliations and trees, Skora's theorem showing when a tree is a leaf space, Otal's compactness theorem, and the special case of the double limit theorem.

There are twelve appendices taking up more than 200 pages. Four appendices begin with a "crash course" on topics readers may have missed in graduate school: commutative algebra (D1.1), covering space theory (D3.1), algebraic number theory (D4.1), and ergodicity (D8.1).

Some of the appendices discuss more or less standard material that is needed in the main text, for instance, amalgamated sums and HNN extensions in Appendix D3 and ends of a topological space in Appendix D5.

Other appendices discuss aspects of Teichmüller theory that are relevant to the hyperbolization theorem without being part of the logical development. For instance, the Margulis lemma is proved in Chapter 11 by a method that works only in dimension 3. A much more general approach to this theorem is given in Appendix D2.

In Appendix D4, we give a deeper introduction to arithmetic Kleinian groups than in Chapter 11, including the relevance of the ideal class group for Bianchi manifolds, examples of arithmetic Kleinian groups with compact quotients, and the construction of Kleinian groups using quaternion algebras.

Appendix D6 discusses the beautiful and surprising result (due to Birman and Series) that on a hyperbolic Riemann surface of finite type, the union of all simple geodesics has Hausdorff dimension 1.

I think of Appendices D7, D8, D9, and D10 as closely related. The main result, due to Masur and Veech, and proved in Appendix D9, is that in each stratum of quadratic differentials on a Riemann surface of finite type, the Teichmüller geodesic flow is ergodic for the measure given by period coordinates. This is obviously a central result of Teichmüller theory.

Of course, the result requires period coordinates, developed in Appendix D7. In Appendix D8, we develop the necessary ergodic theory, in particular Hopf's argument to show that geodesic flow on a surface of negative curvature is ergodic. This is a central result of hyperbolic geometry; we use it to give Mostow's original proof of the Mostow rigidity theorem. But it also gives the plan for proving that the Teichmüller geodesic flow is ergodic, the content of Appendix D9.

To prove this ergodicity result, a first step is to know that the strata have finite period volume. The computation of these volumes has attracted a great deal of work, centered on the contributions of Eskin, Mirzakhani, and Okunkov; we present our approach in Section D9.1. Then to make Hopf's argument work we need to define stable and unstable manifolds for the Teichmüller geodesic flow. This only works for quadratic differentials whose horizontal and vertical flow are *uniquely ergodic*, and further requires that such uniquely ergodic quadratic differentials be of full measure. This finally leads to the desired ergodicity. In Appendix D10, we show that it is possible for horizontal or vertical flow of a quadratic differential to be minimal without being ergodic, something I find surprising even now.

Appendix D11 proves the Sullivan rigidity theorem, which could also be used to prove the hyperbolization theorem.

Finally, in Appendix D12 we construct the Thurston semi-norm on $H^1(M)$ for any 3-manifold M, and show that its unit ball is a finite-sided polyhedron. This is relevant to the hyperbolization of 3-manifolds because the different ways of fibering a 3-manifold over the circle correspond to the integral cohomology classes in the cone over maximal faces of the polyhedron.

Many results in this book apply to the much more general context of *negatively curved spaces*, as defined by Gromov [33]. In this book I take the old-fashioned tack of studying hyperbolic geometry rather than the more general case. This reflects my preference for fine geometry over coarse geometry.

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