

1.7.17. Does the implication “1 \implies 2” in Proposition 1.7.9 remain valid if X, Y are topological spaces?

1.7.18. Let X, Y be metric spaces and let $f : X \rightarrow Y$ be a bijection. Show that the following are equivalent.

1. f is a homeomorphism
2. For all sequences (x_n) in X and all $a \in X$, we have $\lim_{n \rightarrow \infty} x_n = a$ if and only if $\lim_{n \rightarrow \infty} f(x_n) = f(a)$.

1.8 Completeness

Virtually everything in analysis involves constructing objects as limits. Since we usually don't know what the limit is, it is crucial to have some guarantee that limits exist other than checking the definition, which would require knowing the limit. For \mathbb{R} , we have such a guarantee: the axiom of completeness for \mathbb{R} says that every nonempty set of real numbers that is bounded from above has a supremum.

In this section we discuss a similar concept for general metric spaces. Since we cannot talk about a point in a metric space being larger than another point, the concept of upper bound and supremum of a set in an arbitrary metric space does not exist. Thus, we cannot transfer the axiom of completeness verbatim to an abstract metric space. Instead we will look for an equivalent formulation that involves only the metric properties of \mathbb{R} .

The observation that interests us is the following.

Theorem 1.8.1. *The following statements are equivalent:*

1. If $\emptyset \neq A \subset \mathbb{R}$ is bounded, then $\sup A$ exists.
2. Every Cauchy sequence in \mathbb{R} converges to some number in \mathbb{R} .

PROOF. 1 \implies 2: Note that statement 1 implies Theorem 1.7.30. Let (x_n) be a Cauchy sequence in \mathbb{R} . Then (x_n) is bounded by part 1 of Proposition 1.7.17. By Theorem 1.7.29, (x_n) contains a monotone subsequence (x_{n_k}) . Since (x_{n_k}) is a bounded monotone sequence in \mathbb{R} , it must converge by Theorem 1.7.30. Hence, by Proposition 1.7.28, (x_n) itself converges. Thus, 1 \implies 2.

2 \implies 1: Assume statement 2, and let A be a nonempty subset of \mathbb{R} that is bounded from above. We will construct sequences (a_n) and (b_n) of real numbers with the following properties: for all $n \geq 1$,

- a. $a_n \in A$ and b_n is an upper bound for A .
- b. $b_n - a_n \leq \frac{1}{2^{n-1}}(b_1 - a_1)$.
- c. $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$.

Once this has been done, it will follow that A has a least upper bound. To see why this is so, let (a_n) and (b_n) be as above. Given $\epsilon > 0$, it is possible by