Student Solution Manual

Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach, 5th Edition

Errata for the second printing

Notes and errata, complete as of August 7, 2021

You have a copy of the first printing if the copyright page contains, after "Printed in the United States of America" the numbers 10 9 8 7 6 5 4 3 2 1.

You have a copy of the second printing if the numbers end with 2:

10 9 8 7 6 5 4 3 2

Many thanks to Adam Bendorf and Nathaniel Schenker for their contributions to this list.

New listings are marked with red.

Solution 2.10.5 First line:

$$x \leq -1/4, \quad \text{not} \quad x \leq 1/4.$$

Solution 2.10.7 g should be $\mathbf{g}$.

Solutions 2.10.9 and 2.10.15 In keeping with the change to the exercise, F should be $\mathbf{F}$.

Solution 2.3 Part b is wrong; it should be "Statements 1, 2, and 4 are true if $n = m$".

Solution 2.5 To be consistent with notation in the text, all the matrices 0 should be written [0]. In the margin note, note that the two zero matrices on the right of the first displayed equation are of different sizes, as are the two identity matrices; those in the first column are $n - k \times n - k$, while those in the second column are $k \times k$.

Solution 2.7 Part b: In the "final result", the entry in the top row, first column should be $(a + 2)(a - 1)$, not $(a + 2)(a + 1)$.

Solution 2.17 Line 4: "The kernel of T is $\{0\}$", not "The kernel of T is 0".

Solution 2.19 After the first equality, all the f should be af and all the g should be bg.

Solution 2.25 The matrix given by one step of Newton's method has an incorrect entry in the second row, second column. It should be $2 - \frac{1}{12}$, not $2 + \frac{1}{12}$.

Solution 2.37 Line 8: "$q_2(a_1) = \cdots = q_2(b_{k_1}) = 0$" should be "$q_2(a_1) = \cdots = q_2(a_{k_1}) = 0$".

Solution 2.39 Here is a different solution to Exercise 2.39, suggested by Nathaniel Schenker:

Row reduce A by multiplying by elementary matrices on the left. In the matrix BA, where B is a product of elementary matrices, there is a pivotal 1 in every row that is not all zeros, so by column operations (multiplication by elementary matrices D on the right; see Exercise 2.3.11), we can eliminate all the nonzero entries of nonpivotal columns. Now we have a matrix whose only entries are 0's and pivotal 1's. By further multiplication on left and right by elementary matrices we can move all the pivotal 1's to make the matrix of the form J_k. Thus $J_k = \widetilde{B}A\widetilde{D}$, where $\widetilde{B}$ and $\widetilde{D}$ are products of elementary matrices. This gives $A = \widetilde{B}^{-1}J_k\widetilde{D}^{-1}$, or $A = QJ_kP^{-1}$, with $Q = \widetilde{B}^{-1}$ and $P = \widetilde{D}$.

Solution 3.1.25 Part d is not wrong but some readers may find this version clearer:

Saying that the derivative

$$\left[\mathbf{Dg}\begin{pmatrix} u \\ v \end{pmatrix}\right] : \mathbb{R}^2 \to \mathbb{R}^3$$

is injective means that its image is a plane in $\mathbb{R}^3$, not a line or a point. It does not mean that the map $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \left[\mathbf{Dg}\begin{pmatrix} u \\ v \end{pmatrix}\right]$ is injective.

To show that the derivative $\left[\mathbf{Dg}\begin{pmatrix} u \\ v \end{pmatrix}\right]$ is 1–1 at every $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$, it is enough to show that $\left[\mathbf{Dg}\begin{pmatrix} u \\ v \end{pmatrix}\right]\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ implies $a = b = 0$. Write

$$\left[\mathbf{Dg}\begin{pmatrix} u \\ v \end{pmatrix}\right]\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} v\cos uv + 1 \\ 1 \\ v \end{bmatrix} + b\begin{bmatrix} u\cos uv \\ 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The second line tells us that $a = -b$, and the third line says we must have $u = v$. The first line then becomes $u\cos u^2 + 1 = u\cos u^2$, a contradiction, so we must have $a = b = 0$. Thus the derivative of $\mathbf{g}$ is everywhere injective.

Solution 3.3.5 In the third line, $(k, \mathcal{I}_2^0 k)\}$ should be $(k, \mathcal{I}_2^0)\}$, without the extra k. In keeping with the statement of the exercise, all k should be replaced by m.

Nathaniel Schenker suggests a different solution:

Using the "stars and bars" method, consider the problem as counting the ways of allocating m indistinguishable balls to n distinguishable bins. Each configuration of the balls in the bins corresponds to an $I \in \mathcal{I}_n^m$, with each ball in bin j, for $j \in \{1, \dots, n\}$, adding 1 to i_j. Each $I \in \mathcal{I}_n^m$ is represented by a sequence of length $m + n - 1$ containing m stars (representing the m balls) and $n - 1$ bars (which divide the stars into bins). For example, in $\mathcal{I}_4^3$, we represent $(0, 2, 0, 1)$ by $| * *| |*$.

Any choice of positions of the m stars in the sequence automatically implies the positions of the $n-1$ bars, and vice versa. Thus the cardinality of $\mathcal{I}_n^m$ is the number of choices of positions for the stars, that is, $\binom{m+n-1}{m}$, or, equivalently, the number of choices of positions for the bars, that is, $\binom{m+n-1}{n-1}$.

Solution 3.3.9, part d [new, Aug. 7] The existing solution corresponds to an earlier, weaker version of Theorem 3.3.8, which required the stronger condition that crossed partials be continuous. We are replacing it by the following:

d. This does not contradict Theorem 3.3.8 because the first partial derivative $D_2 f$ is not differentiable. The definition of $D_2 f$ being differentiable at $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is that there should exist a line matrix $[a, b]$ such that

$$\lim_{|\vec{\mathbf{h}}| \to 0} \frac{\left| D_2 f(\mathbf{0} + \vec{\mathbf{h}}) - D_2 f(\mathbf{0}) - [a,b]\vec{\mathbf{h}} \right|}{|\vec{\mathbf{h}}|} = 0. \qquad (1)$$

We determine what $[a, b]$ must be by considering $\vec{\mathbf{h}} = \begin{bmatrix} h_1 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{h}} = \begin{bmatrix} 0 \\ h_2 \end{bmatrix}$ (and using the value for $D_2 f$ given in part a):

$$\lim_{h_1 \to 0} \frac{\left| D_2 f \begin{pmatrix} h_1 \\ 0 \end{pmatrix} - D_2 f \begin{pmatrix} 0 \\ 0 \end{pmatrix} - [a,b] \begin{bmatrix} h_1 \\ 0 \end{bmatrix} \right|}{|h_1|} = \frac{|h_1 - a h_1|}{|h_1|}$$

$$\lim_{h_2 \to 0} \frac{\left| D_2 f \begin{pmatrix} 0 \\ h_2 \end{pmatrix} - D_2 f \begin{pmatrix} 0 \\ 0 \end{pmatrix} - [a,b] \begin{bmatrix} 0 \\ h_2 \end{bmatrix} \right|}{|h_2|} = \frac{|b||h_2|}{|h_2|} = 0.$$

To satisfy equation (1), these imply that $[a, b] = [1, 0]$.

But then

$$\lim_{\begin{pmatrix} h \\ h \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{\left| D_2 f \begin{pmatrix} h \\ h \end{pmatrix} - D_2 f \begin{pmatrix} 0 \\ 0 \end{pmatrix} - [1,0] \begin{bmatrix} h \\ h \end{bmatrix} \right|}{\left| \begin{bmatrix} h \\ h \end{bmatrix} \right|} = \frac{\left| \frac{-2h^5}{(2h^2)^2} - h \right|}{\sqrt{2}|h|} = \frac{\frac{3}{2}|h|}{\sqrt{2}|h|} = \frac{3}{2\sqrt{2}} \neq 0.$$

Solution 3.3.15 [new, Aug. 7] Line 3: $\{\, \mathbf{u} \mid \mathbf{f}(\mathbf{u}) = \mathbf{0} \,\}$ (i.e., $\mathbf{u}$, not $\mathbf{x}$).

Line 12: $A^{-1}([\mathbf{Df}(\mathbf{a})]$ should be $A^{-1}(\ker[\mathbf{Df}(\mathbf{a})])$.

Solution 3.5.5 [new, Aug. 7] Nathaniel Schenker points out that in part b, the algebra can be made simpler by using a different change of variables: since $xy + yz = (x + z)y$, we can let $u = (x + z - y)$, so that $x + z = u + y$. Then the quadratic form is

$$(u + y)y = \left(y^2 + uy + \frac{u^2}{4} \right) - \frac{u^2}{4} = \left(y + \frac{u}{2} \right)^2 - \left(\frac{u}{2} \right)^2.$$

Often a clever choice of variables can simplify computations. The advantage of the first solution is that it is a systematic approach that always works: when a quadratic form contains no squares, choose the first term and set the new variable u to be the sum of the variables involved (or, as we did in Example 3.5.7, the difference).

Solution 3.5.7 [new, Aug. 7] Nathaniel Schenker points out that we didn't need to first prove that l must be 0. We have replaced everything up to "In the other direction" by

First, assume Q is a positive definite quadratic form on $\mathbb{R}^n$ with signature (k,l):

$$Q(\mathbf{x}) = \big(\alpha_1(\mathbf{x})\big)^2 + \cdots + \big(\alpha_k(\mathbf{x})\big)^2 - \big(\alpha_{k+1}(\mathbf{x})\big)^2 - \cdots - \big(\alpha_{k+l}(\mathbf{x})\big)^2$$

with the α_i linearly independent.

We want to show that $k = n$ and $l = 0$. Assume $k < n$. Then, by the dimension formula, the $k \times n$ matrix $T = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}$ satisfies $\ker T \neq \{\mathbf{0}\}$. Let $\vec{\mathbf{y}} \neq \mathbf{0}$ be in $\ker T$. Then $Q(\vec{\mathbf{y}}) \leq 0$, so Q is not positive definite. This shows that Q positive definite implies $k \geq n$. Since $k + l \leq n$ with $k \geq 0$, $l \geq 0$, this shows that $k = n$ and $l = 0$.

Solution 3.5.13 [new, Aug. 7] Part b: In line 3, $1/2b_{i,j}$ should be $\frac{1}{2}b_{i,j}$. In line 5, $\vec{\mathbf{x}}^\top C\mathbf{x}$ should be $\vec{\mathbf{x}}^\top C\vec{\mathbf{x}}$.

Solution 3.6.3 [new, Aug. 7] Line 4: $|\mathbf{x}|$ should be $|\vec{\mathbf{x}}|$.

Solution 3.7.3 [new, Aug. 7] We are adding two margins notes, one for part a:

Here it would be easier to find the critical points using a parametrization. In this case the constraints are linear, so finding a parametrization is easy. But usually finding a parametrization requires the implicit function and is computationally daunting, and less straightforward than using Lagrange multipliers (also computationally daunting).

and one for part b:

If you can find critical points, analyzing them using the Hessian matrix approach is systematic.

Solution 3.7.3 [new, Aug. 7] Part b, line 13: $(5/3)^3$ should be $(5/2)^3$.

Solution 3.7.9 [new, Aug. 7] The solution we gave used the directional derivative. We have added a solution using the definition of the derivative:

If A is any square matrix, then

$$\begin{aligned} Q_A(\vec{\mathbf{a}}+\vec{\mathbf{h}}) - Q_A(\vec{\mathbf{a}}) - \vec{\mathbf{h}}^\top A\vec{\mathbf{a}} - \vec{\mathbf{a}}^\top A\vec{\mathbf{h}} &= (\vec{\mathbf{a}}+\vec{\mathbf{h}})^\top A(\vec{\mathbf{a}}+\vec{\mathbf{h}}) - \vec{\mathbf{a}}^\top A\vec{\mathbf{a}} - \vec{\mathbf{h}}^\top A\vec{\mathbf{a}} - \vec{\mathbf{a}}^\top A\vec{\mathbf{h}} \\ &= \vec{\mathbf{a}}^\top A\vec{\mathbf{a}} + \vec{\mathbf{h}}^\top A\vec{\mathbf{a}} + \vec{\mathbf{a}}^\top A\vec{\mathbf{h}} + \vec{\mathbf{h}}^\top A\vec{\mathbf{h}} - \vec{\mathbf{a}}^\top A\vec{\mathbf{a}} - \vec{\mathbf{h}}^\top A\vec{\mathbf{a}} - \vec{\mathbf{a}}^\top A\vec{\mathbf{h}} \\ &= \vec{\mathbf{h}}^\top A\vec{\mathbf{h}}. \end{aligned}$$

So

$$0 \leq \lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{\left|Q_A(\vec{\mathbf{a}}+\vec{\mathbf{h}}) - Q_A(\vec{\mathbf{a}}) - \vec{\mathbf{h}}^\top A\vec{\mathbf{a}} - \vec{\mathbf{a}}^\top A\vec{\mathbf{h}}\right|}{|\vec{\mathbf{h}}|} = \lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{|\vec{\mathbf{h}}^\top A\vec{\mathbf{h}}|}{|\vec{\mathbf{h}}|} \leq \lim_{\vec{\mathbf{h}}\to\vec{\mathbf{0}}} \frac{|\vec{\mathbf{h}}|^2 A}{|\vec{\mathbf{h}}|} = 0$$

Thus the derivative of Q_A at $\vec{\mathbf{a}}$ is the linear function $\vec{\mathbf{h}} \mapsto \vec{\mathbf{h}}^\top A\vec{\mathbf{a}} + \vec{\mathbf{a}}^\top A\vec{\mathbf{h}}$. In our case A is symmetric, so $\vec{\mathbf{h}}^\top A\vec{\mathbf{a}} = \vec{\mathbf{a}}^\top A\vec{\mathbf{h}}$, justifying equation 3.7.52.

Solution 4.1.5 [new, Aug. 7] Part b, third displayed equation: in the first summation, change $i = 0$ to $i = 1$.

Solution 4.1.15 [new, Aug. 7] Part a, case $X \cup Y$: The original solution used Theorem 4.1.21, which assumed that X and Y are disjoint, not necessarily the case here. Here is a new solution:

$X \cup Y$: Whether or not X and Y are disjoint, we have

$$\mathbf{1}_{X \cup Y} \leq \mathbf{1}_X + \mathbf{1}_Y,$$

so

$$\int_{\mathbb{R}^n} \mathbf{1}_{X \cup Y}(\mathbf{x}) |d^n\mathbf{x}| \underset{1}{\leq} \int_{\mathbb{R}^n} \Big(\mathbf{1}_X(\mathbf{x}) + \mathbf{1}_Y(\mathbf{x}) \Big) |d^n\mathbf{x}|$$
$$\underset{2}{=} \int_{\mathbb{R}^n} \mathbf{1}_X(\mathbf{x}) |d^n\mathbf{x}| + \int_{\mathbb{R}^n} \mathbf{1}_Y(\mathbf{x}) |d^n\mathbf{x}|$$
$$\underset{3}{=} 0 + 0 = 0.$$

Case $X \cup Y$: inequality 1 is Proposition 4.1.14, part 3; equality 2 is Proposition 4.1.14, part 1; equality 3: X and Y have volume 0.

One could also argue that $X \cup Y = X \cup (Y - X)$ and apply Theorem 4.1.21 (which concerns disjoint sets), but that requires knowing $Y - X \subset Y$ and $\text{vol}_n Y = 0 \implies \text{vol}_n (Y - X) = 0$, which again requires using $\mathbf{1}_{Y-X} \leq \mathbf{1}_Y$ and applying Proposition 4.1.14, part 3.

Solution 4.2.1[new, Aug. 7] The one error is four lines from the end: 7.5 should be 7. We have expanded on this solution. Here is the new version:

Problem 4.2.1: This kind of question comes up frequently in real life. Some town discovers that 3 times as many children die from leukemia as the national average for towns of that size: 9 deaths rather than 3. This creates an uproar, and an intense hunt for the cause.

But is there really a cause to discover? Among towns of that size, the number of leukemia cases will vary, and some place will have the maximum. The question is: with the number of towns that there are, would you expect to find one that far from the expected value?

4.2.1 The question asked in the exercise is a bit vague: you can't reasonably ask "how likely is this particular outcome". Every individual outcome is very unlikely (though some are more unlikely than others). The question which makes sense is to ask: how many standard deviations is the observed outcome from average? You can also ask whether the number of repetitions of the experiment (in this case 15) is large enough to use the figures of Figure 4.2.5.

The quick answer is that these results are not very likely. If we chart results for each integer, the resulting bar graph does not look like a bell curve, as shown in the margin.

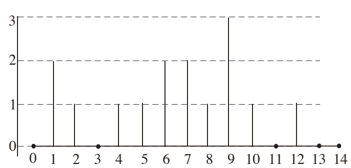

Figure for Solution 4.2.1.

None of the 14 coin tossses resulted in no heads appearing; heads appeared exactly once in two of the 14 toin cosses, and so on.

To give a more detailed answer, we must compute the standard deviation for the experiment. Figuring out all the possible ways of getting all the various totals would be time consuming, so instead we use the fact that if an experiment with standard deviation σ is repeated n times, the central limit theorem asserts that the average is approximately distributed according to the normal distribution with mean E and standard deviation $\sigma/\sqrt{n}$.

The expectation of getting heads when tossing a coin once is 1/2. To compute the variance we consider the two cases. If we get heads, then $(f - E(f))^2 = (1 - 1/2)^2 = 1/4$; if we get tails, we also get

$$(f - E(f))^2 = (0 - 1/2)^2 = 1/4. \quad \text{So}$$

$$\text{Var}(f) = E(f - E(f))^2 = E(1/4 + 1/4) = 1/2(1/2) = 1/4$$

and $\sigma(f) = 1/2$.

So the standard deviation of the average results when tossing the coin 14 times is $\frac{1}{2\sqrt{14}}$. But we are interested in the actual number of heads, not the average. Denote this random variable by T; we multiply the standard deviation of the average results by 14 to get

$$\sigma(T) = \frac{\sqrt{14}}{2} \approx 1.9.$$

But is $n = 14$ large? If we were asked in a court of law, we would not be willing to affirm that a person reporting those results was lying or cheating. If 300 people each tossed a coin 14 times, one of them might very likely come up results more than 2.5 standard deviations from the expected value.

The expectation of T is 7, so (see Figure 4.2.5) for n large we should expect 68% of our results to be between 5.1 and 8.9 heads (within one standard deviation). The actual figure is 5/15 (but if we expanded the range to between 5 and 9, it would be $9/15 \approx 60\%$). For n large we would expect 95% to be within 3.2 and 10.8; the actual figure is $11/15 \approx 73.3\%$.

Solution 4.3.3 [new, Aug. 7] We omitted part b. Here it is:

b. Denote the unit circle by S^1. We have

$$0 \le L_N\left(\mathbf{1}_{S^1}\right) \le U_N\left(\mathbf{1}_{S^1}\right) \le \frac{1}{2^{2N}} 2^N \frac{16}{\sqrt{2}}$$

and

$$\lim_{N\to\infty} \frac{1}{2^{2N}} 2^N \frac{16}{\sqrt{2}} = 0.$$

Solution 4.3.5 [new, Aug. 7] We are replacing the first four lines of the margin note (which referred to $\sin y$, not $\sin(y^2)$) by the following:

Another way to approach this problem is to claim that over a dyadic square $C \in \mathcal{D}_N(\mathbb{R}^2)$ in P that doesn't intersect the boundary of P, the oscillation of $\sin(y^2)$ is strictly less than $2\sqrt{2}/2^N$. Bounding the oscillation of a function f usually involves bounding the derivative of f and applying the mean value theorem. If $f\begin{pmatrix} x \\ y \end{pmatrix} = \sin(y^2)$, then

$$[\mathbf{D}f(\begin{smallmatrix} x \\ y \end{smallmatrix})] = [0,\ 2y\cos(y^2)] \quad \text{and} \quad 0 < 2y\ \cos(y^2) < 2 \text{ for } 0 < y < 1.$$

The above result then follows from Corollary 1.9.2 and the fact that for two points $\mathbf{x}_1, \mathbf{x}_2$ in such a dyadic square,

$$|\mathbf{x}_1 - \mathbf{x}_2| \le \sqrt{2}/2^N.$$

Solution 4.4.5 [new, Aug. 7] We did not define f for $x \in [-1, 0)$ and rational. We are replacing the definition of f by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is rational, written in lowest terms, and } |x| \le 1 \\ 0 & \text{if } x \text{ is irrational, or } |x| > 1, \end{cases}$$

Solution 4.5.5 [new, Aug. 7] Line 6: In the equation for i odd, $\prod_{j=1} k$ should be $\prod_{j=1}^{k}$.

Solution 4.5.19 [new, Aug. 7] In two places, $\int_{-1}^{1}$, not $\int_{1}^{1}$.

Solution 4.7.1 [new, Aug. 7] Paragraph 2, line 2: One reader asked, "Isn't the function given by ye^{-xy} continuous on the boundary of the unit square as well?" Our definition of

$$\int_X f(\mathbf{x})|d^n\mathbf{x}| \quad \text{is} \quad \int_{\mathbb{R}^n} f(\mathbf{x})\mathbf{1}_X(\mathbf{x})|d^n\mathbf{x}|.$$

Thus we have to consider the continuity of $ye^{-xy}\mathbf{1}_P\begin{pmatrix} x \\ y \end{pmatrix}$.

Solution 4.8.11 [new, Aug. 7] To be consistent with the notation introduced on page 462 of the text, $A_{i,1}$ should be $_{[i,1]}$ (three occurrences).

Solution 4.9.3, part b [new, Aug. 7] One reader suggested that we replace "The matrix S ... maps the tetrahedron" by "Multiplication by the matrix S ... maps the tetrahedron". We aren't making that change because we think interchangeably of a matrix as an array of numbers and as the map "multiplication by the matrix".

Solution 4.11.13 [new, Aug. 7] The last line of part a should be

$$|f_k(x_k) - f_\infty(x_k)| = 1 > \epsilon.$$

Solution 5.1.5 [new, Aug. 7] In the three lines after the second displayed equation, an end parenthesis is missing in four places: $\big(tA + (1-t)I$ should be $\big(tA + (1-t)I\big)$.

Nathaniel Schenker suggests a third solution to Exercise 5.1.5:

By the singular value decomposition (Theorem 3.8.1), $T = PDQ^\top$, where P and Q are, respectively $n \times n$ and $k \times k$ orthogonal matrices, and D is an $n \times k$ nonnegative rectangular diagonal matrix. Then

$$T^\top T = QD^\top P^\top PDQ^\top = QD^\top I_n DQ^\top = QD^\top DQ^\top,$$

where the first equality is Theorem 1.2.17 and I_n is the $n \times n$ identity matrix. Then

$$\det(T^\top T) = \det Q \det(D^\top D) \det Q^\top = \det Q \det(D^\top D)(\det Q)^{-1} = \det(D^\top D),$$

where the first equality is Theorem 4.8.4 and the second is Corollary 4.8.5. Since $D^\top D$ is diagonal with nonnegative entries, Theorem 4.8.9 implies that $\det(D^\top D) \geq 0$.

Chapter 5, note on parametrizations Last margin figure, showing a triangle: $r\cos\varphi$, not $r\cos\theta$.

Solution 5.2.7 [new, Aug. 7] Part b, four lines before the end: "with p and q integers satisfying $0 \leq p/q \leq 1$".

Solution 5.3.1 [new, Aug. 7] To be consistent with the statement of the exercise, all the a in part b should be b, and all the t in part c should be b.

Solution 5.3.5 [new, Aug. 7] Part b: In the second displayed equation, dr should be $dr\,d\theta$.

Here is another solution to part b, provided by Nathaniel Schenker and using Definition 5.3.2. In it the elliptic paraboloid is sliced horizontally:

For z fixed, the area of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}\le z$ is $f\begin{pmatrix}x\\y\\z\end{pmatrix}=6\pi z$. Let M denote the z-axis between $z=0$ and $z=a^2$, parametrized as : $t\mapsto\begin{pmatrix}0\\0\\t\end{pmatrix}$, $0\le t\le a^2$. Then the volume of the region in question is

$$\int_M f\begin{pmatrix}x\\y\\z\end{pmatrix}\left|d^1\begin{pmatrix}x\\y\\z\end{pmatrix}\right| = \int_{[0,a^2]} f(\gamma(t))\sqrt{\det([\mathbf{D}\gamma(t)]^\top[\mathbf{D}\gamma(t)]}\,|dt|$$
$$= \int_0^{a^2} 6\pi t\sqrt{1}\,dt = 3\pi a^4$$

Solution 5.3.11 [new, Aug. 7] Line 6 of part b: in the numerator, -4 should be $|-4|$. In the last line, the integral is 2π, not π.

Solution 5.3.13 [new, Aug. 7] Part a: "by equation 5.3.33 and Definition 5.3.1" should be "by equation 5.3.33 and Definition 5.3.1, for any subset $C\subset S_1$ with volume". In the final displayed equation, vol_n should be vol_2. In the last displayed equation, $\mathrm{vol}_2\, S$ should be $\mathrm{vol}_2\, f(C)$. Margin note: Cicero was questor, not governor.

Part b: the 2 in $2(40000)^2$ comes from the fact that there are two polar caps.

Solution 6.1.7 [new, Aug. 7] In part b, we neglected to say that there is 1 elementary 5-form on $\mathbb{R}^5$, since $\binom{5}{5}=\frac{5!}{5!0!}=1$. We also neglected to list the forms:

The 5 elementary 1-forms on $\mathbb{R}^5$ are

$$dx_1,\ dx_2,\ dx_3,\ dx_4,\ dx_5.$$

The 10 elementary 2-forms on $\mathbb{R}^5$ are

$$\begin{aligned}&dx_1\wedge dx_2;\ dx_1\wedge dx_3;\ dx_1\wedge dx_4;\ dx_1\wedge dx_5;\\&dx_2\wedge dx_3;\ dx_2\wedge dx_4;\ dx_2\wedge dx_5;\\&dx_3\wedge dx_4;\ dx_3\wedge dx_5;\\&dx_4\wedge dx_5.\end{aligned}$$

The 10 elementary 3-forms on $\mathbb{R}^5$ are

$$\begin{aligned}&dx_1\wedge dx_2\wedge dx_3;\ dx_1\wedge dx_2\wedge dx_4;\ dx_1\wedge dx_2\wedge dx_5;\\&dx_1\wedge dx_3\wedge dx_4;\ dx_1\wedge dx_3\wedge dx_5;\ dx_1\wedge dx_4\wedge dx_5\\&dx_2\wedge dx_3\wedge dx_4;\ dx_2\wedge dx_3\wedge dx_5;\ dx_2\wedge dx_4\wedge dx_5\\&dx_3\wedge dx_4\wedge dx_5.\end{aligned}$$

The 5 elementary 4-forms on $\mathbb{R}^5$ are

$$dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4;\ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5;\ dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5;$$
$$dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5;\ dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5.$$

The single elementary 5-form on $\mathbb{R}^5$ is

$$dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5.$$

Solution 6.2.7 [new, Aug. 7] In the second displayed formula, $|dt_1 \dots dt_k|$ should be $dt_1 \dots dt_k$. In the paragraph that begins "Indeed", "so" should be "and". In the last displayed equation, and end parenthesis is missing in two places: $\varphi\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)\big)$, not $\varphi\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$.

Solution 6.3.13 [new, Aug. 7] Here is a different solution to part a.

a. Set $v_1(t) = w_2(t) = 1 - t$ and $v_2(t) = w_1(t) = t$ for $0 \le t \le 1$. Then the functions $t \mapsto \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ and $t \mapsto \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$ are linearly independent except at $t = 1/2$ since

$$\det \begin{bmatrix} v_1(t) & w_1(t) \\ v_2(t) & w_2(t) \end{bmatrix} = \det \begin{bmatrix} 1-t & t \\ t & 1-t \end{bmatrix} = (1-t)^2 - t^2 = 1 - 2t;$$

the determinant is 0 only at $t = 1/2$. Thus if we choose $v_3(t)$ and $w_3(t)$ to satisfy

$$v_3(0) = v_3(1) = w_3(0) = w_3(1) = 0 \quad \text{and} \quad v_3(1/2) \neq w_3(1/2)$$

we will be done. One such choice is $v_3(t) = t(1-t)$ and $w_3(t) = 2t(1-t)$:

$$\mathbf{v} = \begin{bmatrix} 1-t \\ t \\ t(1-t) \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} t \\ 1-t \\ 2t(1-t) \end{bmatrix}.$$

Solution 6.5.3 [new, Aug. 7] Part a: This is correct as stated, if $\vec{F}$ is a vector field on $\mathbb{R}^4$. If it is a vector field on $\mathbb{R}^3$, one can either change the flux form to a mass form, getting $M_f\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)\big)$, or change the 3-parallelogram to a 2-parallelogram, getting $\Phi_{\vec{F}}\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)\big)$.

Part c: If f is a function on $\mathbb{R}^2$, this expression makes sense, as a mass form on $\mathbb{R}^2$. If f is a function on $\mathbb{R}^3$, the mass form field is a function of a 3-parallelogram; this should be $M_f\big(P_{\mathbf{x}}(\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}})\big)$ (or $M_f\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)\big)$ or the equivalent).

Part f: This expression should be $\Phi_{\vec{F}} = F_1\, dy \wedge dz - F_2\, dx \wedge dz + F_3\, dx \wedge dy$.

Part e: "This correct" should be "This is correct".

Solution 6.5.7 [new, Aug. 7] Part a: the work is positive if the path points up, and negative if it points down.

Solution 6.5.9 [new, Aug. 7] Third line from the end of part a: $\det[\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}]$ (no comma after $\vec{\mathbf{c}}$)

Solution 6.5.13 [new, Aug. 7] Another solution is to note that $\Phi_{\vec{F}(\mathbf{x})}$, written in coordinates, becomes equation 6.5.25 in the text: a linear combination of elementary $(n-1)$-forms on $\mathbb{R}^n$.

Solution 6.6.1 [new, Aug. 7] Margin note, second paragraph: the condition $x = y = z$ is wrong; it should be $|x| = |y| = |z| = 1$.

Solution 6.6.5 [new, Aug. 7] In parts b–e, ∂X should be $\partial_P X$. Part b, three lines before the end: $v = \sin s\sqrt{6}$ should be $v = \frac{\sin s}{\sqrt{6}}$. Part c: There is no right (or wrong) way to decide whether or not to put arrows on $\gamma(t)$; we are thinking of it both as a point and as a vector.

Solution 6.7.1 [new, Aug. 7] Part b:

$$(-\sin xy\,dx - \sin xy\,dy)$$

should be

$$(-y\sin xy\,dx - x\sin xy\,dy)$$

Solution 6.7.3 [new, Aug. 7] Part a: $\mathbf{d}W_{\vec{F}_2}$ should be $\mathbf{d}\Phi_{\vec{F}_2}$.

Solution 6.7.5 [new, Aug. 7] Solution 6.7.5, part a, line 8 of page 235: "i.e., $h^2 v_{3,3}^2$", not "i.e., $h^2 v_{3,1}^2$". Line 13: each side of the equation should start with a minus sign. Line 19: change $+(2zv_{2,3})$ to $-(2zv_{2,3})$.

Solution 6.7.9 [new, Aug. 7] The solution is incomplete. Since $\mathbf{dd}f = 0$, the complete list of 1-forms satisfying the conditions is larger:

$$\omega = -yz\,dx - xz\,dy + \mathbf{d}f, \quad \text{where } f \text{ is any (at least) } C^2 \text{ function on } \mathbb{R}^3.$$

Solution 6.8.1 [new, Aug. 7] Part a should be:

Numbers: $dx \wedge dy(\vec{\mathbf{v}}, \vec{\mathbf{w}})$, $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$, $\operatorname{grad} f(\mathbf{x}) \cdot \vec{\mathbf{v}}$. Function: $\operatorname{div} \vec{F}$. Vector fields: $\operatorname{grad} f$, $\operatorname{curl} \vec{F}$

Solution 6.8.5 Part d: Note that $\frac{\operatorname{sgn}(x+y+z)}{|x+y+z|}$ can be simplified to $\frac{1}{x+y+z}$. Of course there is no solution to part d if $x + y + z = 0$.

Solution 6.8.9 [new, Aug. 7] Part a: The reference should be to Corollary 3.3.10, not Theorem 3.3.8.

We have changed the statement of part b to

"Show that this is not necessarily true if we only assume that the $D_i f_i$ have partial derivatives everywhere."

The solution is: It is possible for the first partial derivatives to have partial derivatives without being differentiable, and then their crossed partials are not necessarily equal. See Example 3.3.9.

Solution 6.9.3 [new, Aug. 7] See the comments for the exercise on page 645 of the text. We have rewritten the solution, and have added a line (and some explanation) to the equation after equation (3). The new solution is:

6.9.3 a.

$$\begin{aligned}(\mathbf{f}^* W_\xi) P_{\mathbf{x}}(\vec{\mathbf{v}}) &= W_\xi P_{\mathbf{f}(\mathbf{x})}\big([\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}\big) = \xi\big(\mathbf{f}(\mathbf{x})\big) \cdot [\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}} = \big(\xi\big(\mathbf{f}(\mathbf{x})\big)\big)^\top [\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}} \\ &= \Big([\mathbf{Df}(\mathbf{x})]^\top \xi\big(\mathbf{f}(\mathbf{x})\big)\Big)^\top \vec{\mathbf{v}} = \Big([\mathbf{Df}(\mathbf{x})]^\top \xi\big(\mathbf{f}(\mathbf{x})\big)\Big) \cdot \vec{\mathbf{v}} \\ &= W_{[\mathbf{Df}]^\top \xi \circ \mathbf{f}} P_{\mathbf{x}}(\vec{\mathbf{v}}),\end{aligned}$$

so $\mathbf{f}^* W_\xi = W_{[\mathbf{Df}]^\top \xi\circ \mathbf{f}}$.

b. We have $n = m$, so suppose $U \subset \mathbb{R}^n$ is open, $\mathbf{f} : U \to \mathbb{R}^n$ is C^1 and ξ is a vector field on $\mathbb{R}^n$. Then

$$\begin{aligned}
\mathbf{f}^*\Phi_\xi\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_{n-1})\big) &= \Phi_\xi\Big(P_{\mathbf{f}(\mathbf{x})}\big([\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_1,\dots,[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_{n-1}\big)\Big)\\
&= \det\Big[\xi\big(\mathbf{f}(\mathbf{x})\big),[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_1,\dots,[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_{n-1}\Big]\\
&= \det\Big[[\mathbf{Df}(\mathbf{x})][\mathbf{Df}(\mathbf{x})]^{-1}\xi\big(\mathbf{f}(\mathbf{x})\big)[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_1,\dots,[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_{n-1}\Big]\\
&= \det\Big([\mathbf{Df}(\mathbf{x})]\Big[[\mathbf{Df}(\mathbf{x})]^{-1}\xi\big((\mathbf{x})\big),\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_{n-1}\Big]\Big)\\
&= \det[\mathbf{Df}(\mathbf{x})]\det\big[[\mathbf{Df}(\mathbf{x})]^{-1}\xi\big(\mathbf{f}(\mathbf{x})\big),\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_{n-1}\big]\\
&= \Phi_{(\det[\mathbf{Df}])[\mathbf{Df}]^{-1}(\xi\circ\mathbf{f})}\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_{n-1})\big)
\end{aligned}$$

So $\mathbf{f}^*\Phi_\xi = \Phi_{(\det[\mathbf{Df}])[\mathbf{Df}]^{-1}(\xi\circ\mathbf{f})} = \det[\mathbf{Df}]\Phi_{[\mathbf{Df}]^{-1}(\xi\circ\mathbf{f})}$

c. Let $\widetilde{A}_{[i,j]}$ be the matrix obtained from A by replacing the (i,j)th entry by 1, and all other entries of the ith row and jth column by 0. For instance,

$$\text{if}\quad A = \begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}\qquad \text{then}\quad \widetilde{A}_{[3,1]} = \begin{bmatrix} 0 & b_1 & c_1\\ 0 & b_2 & c_2\\ 1 & 0 & 0\end{bmatrix}$$

Note that

$$\det \widetilde{A}_{[i,j]} = (-1)^{i+j}\det A_{[i,j]}. \tag{1}$$

Recall (Exercise 4.8.18) that Cramer's rule says that if $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and if $A_i(\vec{\mathbf{b}})$ denotes the matrix A where the ith column has been replaced by $\vec{\mathbf{b}}$, then $x_i \det A = \det A_i(\vec{\mathbf{b}})$. When A is invertible, so that $\det A \neq 0$, then Cramer's rule says that

$$A\underbrace{\begin{bmatrix}\frac{\det A_1(\vec{\mathbf{b}})}{\det A}\\ \vdots\\ \frac{\det A_n(\vec{\mathbf{b}})}{\det A}\end{bmatrix}}_{\vec{\mathbf{x}}} = \vec{\mathbf{b}},\quad \text{i.e.,}\quad A\begin{bmatrix}\det A_1(\vec{\mathbf{b}})\\ \vdots\\ \det A_n(\vec{\mathbf{b}})\end{bmatrix} = (\det A)\vec{\mathbf{b}}. \tag{2}$$

In the second equation in (2), both sides are continuous functions of A, so the equation is true on the closure of invertible matrices, which is all matrices.

Apply this to $\vec{\mathbf{b}} = \vec{\mathbf{e}}_k$, noting that $\det \widetilde{A}_{[k,i]} = \det A_i(\vec{\mathbf{e}}_k)$. Then Cramer's rule says that

$$A\begin{bmatrix}\det \widetilde{A}_{[k,1]}\\ \vdots\\ \det \widetilde{A}_{[k,n]}\end{bmatrix} = (\det A)\vec{\mathbf{e}}_k,$$

so, using equation 1,

$$A\,\mathrm{adj}(A) = A \begin{bmatrix} \det \widetilde{A}_{[1,1]} & \cdots & \det \widetilde{A}_{[n,1]} \\ \vdots & \dots & \vdots \\ \det \widetilde{A}_{[1,n]} & \cdots & \det \widetilde{A}_{[n,n]} \end{bmatrix} = (\det A)[\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_n] = (\det A)I. \qquad (3)$$

Now we need to show that $\mathbf{f}^*\Phi_\xi = \Phi_{\mathrm{adj}[\mathbf{D}\mathbf{f}](\xi\circ\mathbf{f})}$. For any vector $\xi \in \mathbb{R}^n$ and any invertible $n \times n$ matrix A we have

Equality 1 is the definition of the flux. Equality 2 introduces $AA^{-1} = I$. Equality 3 factors out A. Equality 4 is

$$\det(AB) = \det A \det B.$$

Equality 5 is multilinearity: multiplying one column of a matrix by a number multiplies the determinant by the same number. Equality 6 is $(\det A)A^{-1} = \mathrm{adj}\, A$, which follows from equation 3.

$$\begin{aligned} \Phi_\xi(P_{\mathbf{f}(\mathbf{x})}(A\vec{\mathbf{v}}_1, \dots, A\vec{\mathbf{v}}_{n-1}) &\underset{1}{=} \det[\xi(\mathbf{f}(\mathbf{x})), A\vec{\mathbf{v}}_1, \dots, A\vec{\mathbf{v}}_{n-1}] \\ &\underset{2}{=} \det[AA^{-1}\xi(\mathbf{f}(\mathbf{x})), A\vec{\mathbf{v}}_1, \dots, A\vec{\mathbf{v}}_{n-1}] \\ &\underset{3}{=} \det\Big(A[A^{-1}\xi(\mathbf{f}(\mathbf{x})), \vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}]\Big) \\ &\underset{4}{=} \det A \det[A^{-1}\xi(\mathbf{f}(\mathbf{x})), \vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}] \\ &\underset{5}{=} \det[(\det A)A^{-1}\xi(\mathbf{f}(\mathbf{x})), \vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}] \\ &\underset{6}{=} \det[(\mathrm{adj}(A)\xi(\mathbf{f}(\mathbf{x})), \vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}]. \end{aligned} \qquad (4)$$

and again in both sides the first and last expressions are continuous functions of A, hence valid for all matrices. Applying this to $A = [\mathbf{D}f(\mathbf{x})]$ we find $\mathbf{f}^*\Phi_\xi = \Phi_{\mathrm{adj}[\mathbf{D}\mathbf{f}](\xi\circ\mathbf{f})}$.

Solution 6.10.3 [new, Aug. 7] Equation following "so we may write our integeral as an iterated integral and evaluate": On the left side, $dx_3\,dx_2\,dx_1$, not $|dx_3\,dx_2\,dx_1|$

SOLUTION 6.10.7 [new, Aug. 7] In several places we fail to use arrows to distinguish points from vectors.

Solution 6.10.9 [new, Aug. 7] Second displayed equation: B'' not B".

SOLUTION 6.10.11 [new, Aug. 7] In a number of places we fail to use arrows to distinguish points from vectors.

Solution A1.5 Part c: Four lines before the end of the page, "at most $c(b/a)^k$" should be "at most $c(b/c)^k$".