Appendix D9

Teichmüller flow is ergodic

This appendix proves Theorem D9.0.1, due to Masur [54] and Veech [88]. The proof applies Hopf's argument to Teichmüller flow on moduli space, and even to Teichmüller flow on each component of each *stratum* of the bundle of quadratic differentials over moduli space. Teichmüller flow and strata of the space of quadratic differentials are discussed in Appendix C5 in Volume 2.

Let S be a compact oriented surface, and let $P \subset S$ be a finite set; let $\mathcal{Q}_{S,P}$ be the set of pairs $((X,\varphi),q)$ with $\varphi : (S,P) \to X$ a marking of the Riemann surface X and q an element of $Q^1(X-\varphi(P))$; recall that $\mathcal{Q}_{S,P}$ is the cotangent bundle of $\mathcal{T}_{S,P}$. The quadratic differential q actually contains all the information of $((X,\varphi),q)$, and we will refer to elements of $\mathcal{Q}_{S,P}$ as simply q. The set of quadratic differentials $q \in \mathcal{Q}_{S,P}$ with zeros and poles having the same orders forms a *stratum* of $\mathcal{Q}_{S,P}$. Every $q \in \mathcal{Q}_{S,P}$ is contained in a unique connected component of some stratum.

Theorem D9.0.1 *On each connected component of a stratum of $\mathcal{Q}_{S,P}$ the Teichmüller flow is ergodic.*

I thank Curt McMullen for suggesting (and insisting) that I include this material, since it has been central to much of the work in Teichmüller theory in recent years.

The underlying idea is straightforward. Set

$$q_t := \Big(e^{t/2}|\operatorname{Re}\sqrt{q}| + ie^{-t/2}|\operatorname{Im}\sqrt{q}|\Big)^2. \qquad \text{D9.0.1}$$

The quadratic differential q_t is holomorphic on a different surface: if q is holomorphic on X, and if $z = x + iy$ is a local coordinate on X for which $q = dz^2$, then $e^{t/2}x + ie^{-t/2}y$ is a local coordinate on the surface X_t on which q_t is holomorphic.

The *Teichmüller geodesic* through $q \in \mathcal{Q}_{S,P}$ is the parametrized curve

$$t \mapsto q_t; \qquad \text{D9.0.2}$$

this is a parametrization by arc length. The Teichmüller distance between $\varphi : S \to X$ and $\varphi : S \to X_t$ is t.

The geodesic $t \mapsto q_t$ stays in the stratum of q.

The Teichmüller geodesic stretches the horizontal foliation of q and compresses the vertical foliation. In that sense, it "goes to" the point at infinity corresponding to the horizontal foliation, and "comes from" the point at infinity corresponding to the vertical foliation. Thus each stratum is filled with stable and unstable manifolds of points at infinity for the Teichmüller flow. Two quadratic differentials q_1 and q_2 should belong to the same stable manifold if $|\operatorname{Re}\sqrt{q_1}| = |\operatorname{Re}\sqrt{q_2}|$, i.e., if their vertical foliations coincide. Similarly, they should belong to the same unstable manifold if their horizontal foliations coincide. Hopf's argument (Theorem D8.3.2) then shows that for any continuous function with compact support in a stratum, the forward and backwards ergodic averages are constant.

The devil is in the details. One of these "details" is showing that strata have finite volume for the period measure; the ergodic theorem (Theorem D8.1.4) applies only to finite measures (as written, it applies to probability measures). This is not a small issue; computing the volumes of these stata has required the combined efforts of a stellar constellation of mathematicians: Masur, Veech, Kontsevich, Zorich, Eskin, Okunkov, Mirzakhani, Athreya, Goujard, and many others. The result is contained in [31] and [6].

This appendix is divided into three parts. The first consists of Section D9.1, which shows that unit strata have finite volume. The second consists of three sections, all concerned with unique ergodicity. Section D9.2 defines unique ergodicity, gives examples, and ends with the statement that if a horizontal foliation is uniquely ergodic, every trajectory is *typical*. Section D9.3 shows that quadratic differentials with uniquely ergodic horizontal foliations have full measure. Section D9.4 shows that two uniquely ergodic foliations belonging to the same stable manifold are asymptotic. Finally, in Section D9.5, we show how to use Hopf's argument to complete the proof that Teichmüller flow is ergodic.

D9.1 Unit strata have finite volume

The ergodic theorems, both for flows (Theorem D8.1.4) and for maps (Theorem D8.1.2) apply to probability spaces and, more generally, to spaces of finite measure, but not to spaces of infinite measure. Ergodic theory doesn't make sense in spaces of infinite measure. The main result of this section is Theorem D9.1.2, which says that each *unit stratum* has finite *period volume*.

When I discussed strata in Volume 2 (see the subsection "Strata in $\mathcal{Q}_S$ and $\mathcal{C}_S$" in Appendix C5), I used the standard definition of a stratum, given by the orders of the zeros and poles. This is unfortunate: different components of a stratum may have different dimensions, so there is no such thing as the dimension of a stratum; see Figure D9.1.1. Instead, a stratum should be defined as a *connected component* of the space of quadratic differentials having zeros and (simple) poles of specified orders. We classify such connected components as *Abelian* or *non-Abelian*.