Appendix D8
Ergodic flow, Hopf's argument, and Mostow rigidity

In this appendix we give the original proof of Mostow's rigidity theorem [64], [65]. It isn't easier or shorter than the one given in Chapter 12, but it brings up the importance of knowing that a Kleinian group acts *ergodically* on the limit set, an important statement in its own right. It also provides an excuse for introducing the ergodic theorem and Hopf's ergodicity argument. First we restate Mostow's rigidity theorem:

Theorem 12.4.1 (Mostow's rigidity theorem) *Let M and N be compact hyperbolic 3-manifolds, and let $f : M \to N$ be a homotopy equivalence. Then f is homotopic to a unique isometry.*

The key step in both proofs is that a Beltrami form on $\mathbb{P}^1$ invariant under the fundamental group of a hyperbolic 3-manifold M of finite volume vanishes. In Chapter 12 we proved this using McMullen's rigidity theorem (Theorem 12.3.1), based on the Lebersgue density theorem. Here we use Corollary D8.3.4, based on the ergodic theorem for flows.[1]

The ergodic theorem for maps and the ergodic theorem for flows are central results of measure theory. I believe both theorems, as well as the Poincaré recurrence theorem, should be part of a first-year graduate course in real analysis. Since they are not, I provide proofs in Section D8.1.

Ergodic theory is concerned with measure-preserving maps and flows. Classical mechanics provides an essential source of such flows (and Poincaré sections of Hamiltonian flows are an essential source of such maps): Hamiltonian flows and the special case of geodesic flows preserve the *Hamiltonian measure*. In Section D8.2 we discuss geodesic flows and Hamiltonian flows.

The ergodicity of Hamiltonian flows is of central interest in classical mechanics. Eberhardt Hopf found an argument that sometimes proves ergodicity; in particular it proves the ergodicity of the geodesic flow on compact manifolds of negative curvature, including hyperbolic manifolds. We discuss *Hopf's argument* in Section D8.3.

[1] Reminiscing in the *Yale News* in 2013, Mostow recalled the moment he suddenly thought, "Use ergodicity! ... The final idea jumped out at me as I was waiting in my car at a red light at the corner of Whalley Avenue and Fitch Street. I get a high every time that I pass that intersection."

Many (perhaps most) proofs that a transformation is ergodic consist of appropriate modifications of Hopf's idea. Crucially for us, Appendix D9 on Teichmüller geodesic flow is a delicate application of Hopf's idea.

D8.1 Ergodicity: a crash course

For those who need it, here we will briefly discuss ergodicity and state and prove the ergodic theorem for maps, the ergodic theorem for flows, and the Poincaré recurrence theorem.

Some basic definitions

A *measure space* is a set X together with

- a σ-algebra $\mathcal{A}$ of subsets of X
- a function $\mu\colon \mathcal{A} \to [0, \infty]$ that is σ-additive: if $i \mapsto A_i$ is a sequence of mutually disjoint elements of $\mathcal{A}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

A *probability space* is a measure space X such that $\mu(X) = 1$.

Elements of $\mathcal{A}$ are called *measurable sets* (more precisely, μ-measurable). If $\mathcal{A}$ is a σ-algebra of subsets of X, and $\mathcal{B}$ is a σ-algebra of subsets of Y, then $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is *measurable* if for all $B \in \mathcal{B}$, we have $f^{-1}(B) \in \mathcal{A}$.

If μ is a measure on $(X, \mathcal{A})$, then $f_*\mu$ (the *push forward of* μ *by* f) is the measure on $(Y, \mathcal{B})$ defined by $f_*\mu(\mathcal{B}) = \mu(f^{-1}(\mathcal{B}))$.

In geometrical applications the space X is often a topological space as well as a measure space and $\mathcal{A}$ is the Borel σ-algebra; references to $\mathcal{A}$ are often omitted. In particular, $\mathbb{R}$ is assumed to carry the Borel σ-algebra unless explicitly stated otherwise.

Probabilists need to be more careful than the geometers: defining conditional probability involves σ-algebras. The proof of the ergodic theorem borrowed from [45] is due the probabilist Jacques Neveu and uses different σ-algebras in a crucial way, so we will be careful about σ-algebras.

A measurable map $\Phi : (X, \mathcal{A}, \mu) \to (Y, \mathcal{B}, \nu)$ of measure spaces is *measure preserving* if $\Phi_*\mu = \nu$. If the domain and codomain of Φ are the same, then μ is called an *invariant measure under* Φ.

Below $\mathbb{R} \times X$ carries the σ-algebra generated by sets $B \times A$ with $A \in \mathcal{A}$ and $B \subset \mathbb{R}$ Borel. A *flow* on X is a measurable map $\varphi : \mathbb{R} \times X \to X$ such that $\varphi(0, x) = x$ for all $x \in X$ and $\varphi(t_1 + t_2, x) = \varphi(t_2, (\varphi(t_1, x))$ for all $t_1, t_2 \in \mathbb{R}$. The flow φ is *measure preserving* if it is measurable and for all $t \in \mathbb{R}$, the map $x \mapsto \varphi(t, x)$ is measure preserving.

Ergodic theory is concerned with measure-preserving maps and flows.