Appendix D7
Period coordinates

Appendix C5 in Volume 2 constructed local coordinates on strata in the bundle $\mathcal{C}_S$ of Abelian differentials above Teichmüller space. At the time I said that the arguments go through with modifications for the bundle $\mathcal{Q}_S$ over Teichmüller space. Here I come clean about just how that works.

Let S be a compact surface of genus g and let $P \subset S$ be a finite set with n points. Then by Proposition 6.6.2 the space of pairs

$$\mathcal{Q}_{S,P} := \Big\{ \Big(\tau \in \mathcal{T}_{S,P} \text{ represented by } \varphi : S \to X \Big), \Big(q \in Q^1(X - \varphi(P)) \Big) \Big\}$$

is the cotangent bundle of $\mathcal{T}_{S,P}$; the natural projection

$$\eta_{S,P} : \mathcal{Q}_{S,P} \to \mathcal{T}_{S,P} \qquad \text{D7.1}$$

is projection on the first coordinate.

The Teichmüller space $\mathcal{T}_{S,P}$ has dimension $3g - 3 + n$. Because $\mathcal{Q}_{S,P}$ is its cotangent bundle, $\mathcal{Q}_{S,P}$ is a complex manifold of dimension $6g - 6 + 2n$. It is easy to give coordinates on the cotangent bundle of any manifold if you know coordinates on the manifold. Let $U \subset \mathcal{T}_{S,P}$ be open, and let $\varphi : U \to \mathbb{C}^k$ be a local coordinate (i.e., an analytic map that is a homeomorphism to its image V). Then the map $\Phi : V \times \mathbb{C}^k \to \eta_{S,P}^{-1}(U)$ given by

$$\Phi(v, \mathbf{a}) = \Big(\varphi^{-1}(v), a_1 d\varphi_1 + \cdots + a_k \, d\varphi_k \Big) \qquad \text{D7.2}$$

is a chart of $\mathcal{Q}_{S,P}$.

The problem with this description is that it makes no reference to the geometry that the Riemann surface X acquires from q: area, lengths of curves, etc. In this appendix we describe how to use such geometric properties to put coordinates on $\mathcal{Q}_{S,P}$. These coordinates are called *period coordinates*. They were first used in [22] and further developed in [41].

There is a fly in the ointment. If q has multiple zeros, these coordinates do not allow for break-up of the zeros, so near such q, the period coordinates are only local coordinates on the *stratum* containing q: those quadratic differentials (on varying Riemann surfaces) that have zeros and poles with the same multiplicities as those of q.

Remark It is possible to extend the notion of period coordinates to allow for break-up of zeros, but it involves hypercohomology (see [41]). Here we do not want to use anything so elaborate. △

Remark The words *Riemann surface* and *non-singular complex curve*, often abbreviated to *curve*, are synonymous. Analysts tend to speak of Riemann surfaces, and algebraic geometers of curves, i.e., one-dimensional complex varieties. Analysts don't have a word for singular curves; the expression "Riemann surface with singularities" does not roll off the tongue. Here we need to consider singular curves; we will use both languages. △

Period coordinates and the Riemann surface $\widehat{\widehat{X}}_q$

Period coordinates on the cotangent bundle $\mathcal{Q}_{S,P}$ are functions of the form $\int_\gamma \sqrt{q}$, where γ is a (real) curve on the underlying Riemann surface X. To define period coordinates with precision, we need a Riemann surface on which q has a square root, i.e., a Riemann surface that carries a 1-form ω_q satisfying $\omega_q^2 = q$. The "desingularized" Riemann surface $\widehat{\widehat{X}}_q$ is our answer to this problem. See Figure D7.1.

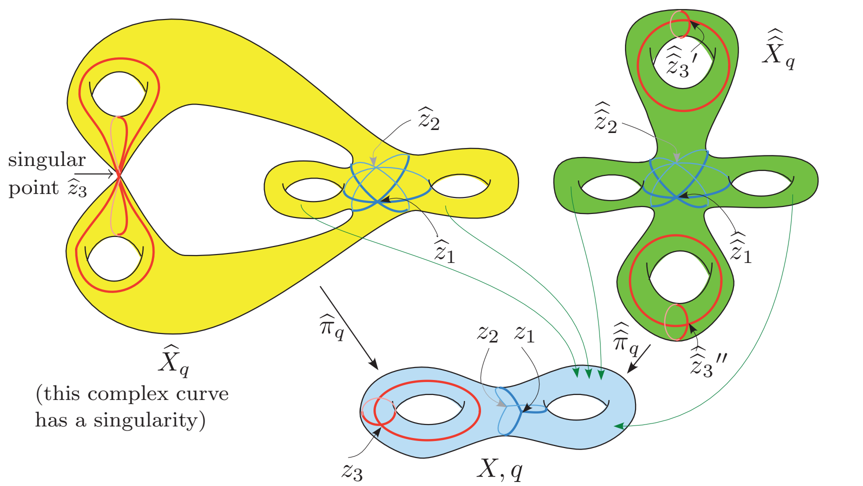

Figure D7.1 At bottom, a surface X (blue) with a quadratic differential q. The quadratic differential q has closed horizontal leaves; the critical graph (blue and red) breaks X into two cylinders. The critical graph has two simple zeros (the 3-pronged points z_1 and z_2) and one double zero (the 4-pronged point z_3). These three points form Z_q. The double zero creates a singular point in $\widehat{X}_q$ (yellow), so $\widehat{X}_q$ is not a Riemann surface. This singular point $\widehat{z}_3$ corresponds to two ends of $\widehat{X}_q - \widehat{Z}_q$. The endpoint compactification gives $\widehat{\widehat{X}}_q$ (green) at right; the singular point gives rise to two simple zeros of ω_q, the points $\widehat{\widehat{z}}_3{}'$ and $\widehat{\widehat{z}}_3{}''$. The simple zeros of q give rise to double zeros of ω_q, the points $\widehat{\widehat{z}}_1$ and $\widehat{\widehat{z}}_2$.

Let $q \in Q^1(X - \varphi(P))$ be an integrable nonzero quadratic differential; let Z_q be the set of zeros of q. A first stab at "the Riemann surface of $\sqrt{q}$"