Appendix D6
The space of simple geodesics

In this appendix we prove that if X is a hyperbolic surface of finite area, the union of all simple geodesics on X has *Hausdorff dimension 1*.

Definition D6.1 (Hausdorff measure, Hausdorff dimension) Let M be a metric space. Set

$$\mu_d(M) := \lim_{r \to 0} \inf \sum r_i^d, \qquad \text{D6.1}$$

with the infimum taken over all covers of M by balls B_i of radius r_i with $0 < r_i \leq r$. We call $\mu_d(M)$ the *Hausdorff measure of M in dimension d*. The *Hausdorff dimension* of M, denoted $\mathrm{Hdim}(M)$, is that number D such that for all $d > D$, we have $\mu_d(M) = 0$, and for all $d < D$, we have $\mu_d(M) = \infty$.

Definition D6.1 is illustrated by Figure D6.1.

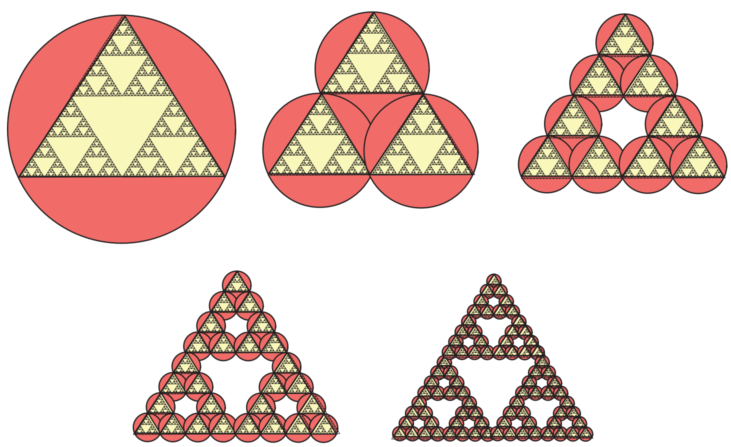

FIGURE D6.1 A Sierpinsky gasket covered by one ball of radius 1, three balls of radius 1/2, nine balls of radius 1/4, and more generally 3^n balls of radius $1/2^n$. For each n this realizes the most efficient cover by balls of radius $\leq 1/2^n$. We have

$$\lim_{n \to \infty} 3^n \left(\frac{1}{2^n}\right)^d = \begin{cases} 0 & \text{if } d > \frac{\ln 3}{\ln 2} \\ \infty & \text{if } d < \frac{\ln 3}{\ln 2} \end{cases}$$

So the Sierpinski gasket has Hausdorff dimension $\ln 3 / \ln 2$.

Exercise D6.2 Show that for any metric space, such a number D exists in $[0, \infty]$, so $\mathrm{Hdim}(M)$ is the infimum of the d such that $\mu_d(M) = 0$. $\diamondsuit$

Exercise D6.3 Show that if M is a metric space, and $M' \subset M$ carries the induced metric, then $\mathrm{Hdim}\, M' \leq \mathrm{Hdim}\, M$. $\diamondsuit$

Lemma D6.4 *A countable union of sets of Hausdorff dimension D has Hausdorff dimension D.*

PROOF Let the sets be $M_1, M_2, \ldots$. Suppose $d > D$, and choose $\epsilon > 0$. For any $r > 0$, there exists a cover of M_i by balls $B_{i,j}$ of radius $r_{i,j} < r$ with $\sum_j r_{i,j}^d < \epsilon/2^i$. Then the union of all the balls $B_{i,j}$ covers $\cup_i M_i$, and

$$\sum_i \sum_j r_{i,j}^d < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon. \qquad \square$$

Exercise D6.5 Show that any nonempty open subset of $\mathbb{R}^n$ has Hausdorff dimension n. $\diamondsuit$

Theorem D6.6 is due to Joan Birman and Caroline Series [8].

Theorem D6.6 *The union Z of all simple geodesics on a complete hyperbolic surface X of finite area has Hausdorff dimension 1.*

PROOF The proof will take the rest of this appendix and will require two further propositions. Let us first assume that X is compact. Choose a maximal multicurve Γ on X, cutting X into trousers, which we will take to be closed. Then simple geodesics on X come in three flavors: simple closed geodesics, infinite geodesics that at one end or the other (or both) spiral towards an element of Γ, and Γ-biinfinite geodesics that in both directions cross curves of Γ infinitely often.

For the first class, there are countably many simple geodesics, so (by Lemma D6.4 and Exercise D6.5) the set of simple closed geodesics has Hausdorff dimension 1.

We deal with the second class by reducing it to the third. Choose three maximal multicurves Γ_1, Γ_2, and Γ_3 that have no curves in common; further for later purposes we will need to choose Γ_1, Γ_2, and Γ_3 such that the trousers each one defines have distinct boundary components. Such multicurves exist; see Figure D6.2.

Then any infinite geodesic can have ends asymptotic to curves of only two of them, and will necessarily be biinfinite with respect to the third. Thus it is enough to show that the set of Γ-biinfinite simple geodesics with respect to one maximal multicurve has dimension 1.

To prove Theorem D6.6, we will require a statement from hyperbolic geometry. Proposition D6.7 says that if geodesics join points of geodesics that are far apart, then their central parts are very close together. See Figure D6.3.