Appendix D5
Ends of a topological space

Intuitively, an *end* of a topological space is a way of going to infinity. Ends of a topological space X were defined by Hans Freudenthal [28, 29, 30]; John Stallings [79, 80] also made essential contributions. Most of the older literature requires that X be connected, locally connected, and second countable, but recently Nyikos [67] and Baillif, Fernández-Bretón, and Vlamis [26] have shown that ends of spaces that are not second countable are useful in classifying non-metrizable manifolds like the one discussed in Example 1.3.1 in Volume 1. Theorem D5.11 is proved in this generality, and the author thanks Vlamis for his help in writing this appendix.

Inverse limits

Inverse limits, also called *projective limits* (or just *limits* by category theorists) are a standard construction in many parts of mathematics. We will describe them in any category, speaking of *objects* and *morphisms*. The category relevant to this appendix is the topological category: for "object" substitute "topological space" and for "morphism" substitute "continuous map".

Inverse limits are defined for objects and morphisms labeled by a *partially ordered set*, called a *poset*. A poset $(\mathcal{E}, \preceq)$ is *directed* if for all $\alpha, \beta \in \mathcal{E}$ there exists $\gamma \in \mathcal{E}$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Inverse limits are well defined only when the labeling poset is directed.

Definition D5.1 (Cofinal subset) A subset $\mathcal{F}$ of a poset $\mathcal{E}$ is *cofinal* if for all $\alpha \in \mathcal{E}$ there exists $\beta \in \mathcal{F}$ such that $\alpha \preceq \beta$.

In practice (we will see several examples), it often happens that $\mathcal{E}$ is huge and incomprehensible and that a cofinal subset $\mathcal{F}$ is far more amenable.

An *inverse system* labeled by a directed poset $\mathcal{E}$ is a collection of objects X_α, $\alpha \in \mathcal{E}$ together with morphisms $\rho^\alpha_\beta : X_\beta \to X_\alpha$ for all $\alpha, \beta \in \mathcal{E}$ with $\alpha \preceq \beta$ satisfying

$$\rho^\alpha_\beta \circ \rho^\beta_\gamma = \rho^\alpha_\gamma \quad \text{and} \quad \rho^\alpha_\alpha = \text{id}. \tag{D5.1}$$

The inverse limit $\varprojlim\limits_{\alpha \in \mathcal{E}, \preceq} X_\alpha$ is defined by the following universal property.

Definition D5.2 (Universal property of inverse limit) Let

$$\left\{(X_\alpha, \rho^\alpha_\beta),\ \alpha, \beta \in \mathcal{E},\ \alpha \preceq \beta\right\} \tag{D5.2}$$

be an inverse system labeled by the directed poset $(\mathcal{E}, \preceq)$. For every $\beta \in \mathcal{E}$, there is a morphism

$$\pi^\beta : \varprojlim_{\alpha \in \mathcal{E}, \preceq} X_\alpha \to X_\beta \tag{D5.3}$$

satisfying

$$\rho^\alpha_\beta \circ \pi^\beta = \pi^\alpha \quad \text{when} \quad \alpha \preceq \beta. \tag{D5.4}$$

Moreover, if Z is an object, and for every $\alpha \in \mathcal{E}$ there is morphism $p^\alpha : Z \to X_\alpha$ such that

$$\rho^\alpha_\beta \circ p^\beta = p^\alpha \quad \text{when } \alpha \preceq \beta, \tag{D5.5}$$

then there exists a unique morphism

$$p : Z \to \varprojlim_{\alpha \in \mathcal{E}, \preceq} X_\alpha \tag{D5.6}$$

such that $p^\alpha = p \circ \pi^\alpha$ for all $\alpha \in \mathcal{E}$.

By Yoneda's lemma, this universal property defines the inverse limit (if it exists) up to unique isomorphism. It exists in any category admitting products: there is a natural inclusion

$$\varprojlim_{\alpha \in \mathcal{E}, \preceq} X_\alpha \subset \prod_{\alpha \in \mathcal{E}} X_\alpha. \tag{D5.7}$$

Exercise D5.3 Show that

$$\varprojlim_{\alpha \in \mathcal{E}, \preceq} X_\alpha = \left\{ (x_\alpha)_{\alpha \in \mathcal{E}} \in \prod_{\alpha \in \mathcal{E}} X_\alpha \ \middle|\ \rho^\alpha_\beta x_\beta = x_\alpha \right\}. \qquad \diamondsuit$$

In computing inverse limits, it is often useful to restrict to a cofinal set.

Proposition D5.4 *If $(\mathcal{E}, \preceq)$ is a directed poset and $\mathcal{F} \subset \mathcal{E}$ is cofinal, then the projection $\prod_{\alpha \in \mathcal{E}} X_\alpha \to \prod_{\beta \in \mathcal{F}} X_\beta$ induces a isomorphism*

$$\varprojlim_{\alpha \in \mathcal{E}} X_\alpha \xrightarrow{\sim} \varprojlim_{\beta \in \mathcal{F}} X_\beta. \tag{D5.8}$$

(In the rest of this appendix we will write this canonical isomorphism as an equality.)

Exercise D5.5 Prove Proposition D5.4. $\diamondsuit$