Appendix D2
The Margulis lemma: another proof

We proved the Margulis lemma in Section 11.8; here we give a different proof. First we restate the theorem.

Theorem D2.1 (Margulis lemma restated) *There exists a number* $r_0 > 0$ *such that for any torsion-free discrete subgroup* $\Gamma \subset \operatorname{Aut} \mathbb{H}^3$ *and any point* $\mathbf{p} \in \mathbb{H}^3$*, the elements* $\gamma \in \Gamma$ *such that* $d(\mathbf{p}, \gamma(\mathbf{p})) < r_0$ *generate an elementary group.*

The proof here is not simpler or shorter, but I feel that it is more natural. Certainly it works far more generally: it works for hyperbolic spaces of any dimension, whereas the proof in Section 11.8 uses Jorgensen's inequality and so works only for $\mathrm{PSL}_2\, \mathbb{C}$. But we will restrict the proof to $\mathbb{H}^3$, since we haven't developed the necessary preliminaries for the general case.

The idea of the proof is simple. If G is a Lie group, then the map $G \times G \to G$ given by $(f, g) \mapsto [f, g]$ fixes the identity, and its Taylor series at the identity starts with quadratic terms. Hence there is a neighborhood U of the identity such that the sequence

$$U_0 = U, \;\; U_1 = [U, U], \;\; U_2 = [U, [U, U]], \;\; \dots \qquad \text{D2.1}$$

is a decreasing sequence with intersection reduced to the identity. If $\Gamma \subset G$ is discrete, then there exists n such that all n-fold brackets of elements of $\Gamma \cap U$ are the identity. It isn't a big step to see that the elements of $\Gamma \cap U$ generate a nilpotent group.

Of course, the Margulis lemma is not about the subgroup of Γ generated by the elements close to the identity: it is about those that move a particular point $\mathbf{p} \in \mathbb{H}^3$ a small amount. These can include elliptic elements far from the identity. Thus it concerns the subgroup generated by elements of Γ close to the stabilizer $\mathrm{Stab}(\mathbf{p})$ of $\mathbf{p}$, which is a compact subgroup. Most of the proof is reducing a neighborhood of $\mathrm{Stab}(\mathbf{p})$ to a neighborhood of the identity.

Three preliminary lemmas

We will require several lemmas, each interesting in its own right. A group G is *solvable* if there exists a finite chain of subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_k = \{1\} \qquad \text{D2.2}$$

such that each G_{i+1} is normal in G_i and G_i/G_{i+1} is commutative.

Lemma D2.2 *Every torsion-free solvable Kleinian group $G \subset \mathrm{Aut}(\mathbb{H}^3)$ is elementary.*

Proof We will work by induction on the length of the chain of subgroups $G \supset [G, G] \supset [[G, G], [G, G]] \supset \cdots \supset \{I\}$; by the inductive hypothesis, we can assume $H := [G, G]$ is elementary. Therefore the limit set Λ_H of H consists of one or two points, and these are the only finite orbits of H in $\partial\overline{\mathbb{H}}^3$. For any $g \in G$ and $h \in H$, we have $g^{-1}hg \in H$, hence $g^{-1}hg\Lambda_H = \Lambda_H$, hence $h(g\Lambda_H) = g(\Lambda_H)$. In particular, $g(\Lambda_H)$ is also invariant under H, hence equal to Λ_H.

This says that Λ_H is a closed subset of $\partial\overline{\mathbb{H}}^3$, invariant under G, which therefore contains Λ_G (in fact $\Lambda_H = \Lambda_G$). Thus G is elementary. □

Lemma D2.3 *In any Lie group $\mathcal{G}$, there exists a neighborhood U of the identity element I such that any discrete subgroup Γ of $\mathcal{G}$ generated by $\Gamma \cap U$ is nilpotent. In particular Γ is solvable.*

Proof Consider commutation as a map from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$ taking (x, y) to $[x, y] := xyx^{-1}y^{-1}$, and consider the Taylor expansion of this map at the point (I, I). The map is constant on both $\mathcal{G} \times \{I\}$ and $\{I\} \times \mathcal{G}$, so there are no linear terms. Once a norm is fixed on the Lie algebra of $\mathcal{G}$, and hence on any neighborhood V of the identity for which the exponential map is a diffeomorphism (and which has compact closure), there is a constant $C \geq 1$ such that $|[x, y]| \leq C|x||y|$ for all $x, y \in V$. Now choose $U \subset V$ such that $|x| < 1/(2C)$ for all $x \in U$, and such that $U^{-1} = U$.

Let $S_0 = S_0^{-1}$ denote the elements of Γ that are contained in U, and define S_n recursively to be the set of commutators $ghg^{-1}h^{-1}$ with $g \in S_0$ and $h \in S_{n-1}$. It is easy to prove by induction that the elements of S_n have norm less than $1/(2^{n+1}C)$. Hence, for sufficiently large n, we have $S_n = I$. Applying the identity $[x, yz] = [x, y][[y, [x, z]][x, z]$ repeatedly, we can prove that any n-fold commutator formed from finite products of elements of S_0 is a product of m-fold commutators of elements of S_0, with $m \geq n$, hence equals the identity. Thus the lower central series of Γ terminates, establishing that Γ is nilpotent. □

Lemma D2.4 *If G is a Kleinian group and $H \subset G$ is a subgroup of finite index, then their limit sets are equal: $\Lambda_H = \Lambda_G$. In particular, any group that contains an elementary subgroup of finite index is itself elementary.*

Proof The inclusion $\Lambda_H \subset \Lambda_G$ is immediate from the definition of the limit set. It is easy to see that Λ_G is the union of the accumulation points of orbits of a point in $\mathbb{H}^3$ under each of the right cosets of H in G, so Λ_G is a finite union of translates of Λ_H.

Thus if $\Lambda_H = \emptyset$, then $\Lambda_G = \emptyset$ and the result is true in that case.