Appendix D12
Fibrations and the Thurston norm

It may seem surprising that a 3-manifold should fiber over the circle. These fibrations are hard to visualize: consider how difficult it is to construct the fibration of the complement of the figure-eight knot (Example 13.1.3). My geometric instinct leads me to expect it to be rare for a manifold M to fiber over the circle, and much rarer for it to fiber over the circle in many different ways.

I was therefore surprised to learn, in 2014 or 2015, that this is wrong as soon as the dimension of $H^1(M;\mathbb{R})$ is greater than 1; then if M fibers over the circle in one way, it fibers over the circle in lots of different ways. Moreover, M need not be 3-dimensional; the result is valid in any dimension! I was surprised again, in fall 2017, to learn from a talk by Chenxi Wu that the requirement $\dim H^1(M;\mathbb{R}) > 1$ is not as innocuous as one might expect. Any manifold that fibers over the circle with fiber S can be written

$$M = \big(S \times [0,1]\big)/(x,0) \sim (f(x),1), \tag{D12.1}$$

i.e., M is the mapping cylinder of a homeomorphism $f : S \to S$. The cohomology of M can be computed by the Mayer-Vietoris theorem, and we find that $\dim H^1(M;\mathbb{R}) > 1$ if and only if

$$\ker(f^* - \mathrm{id}) : H^1(S,\mathbb{R}) \to H^1(S,\mathbb{R}) \tag{D12.2}$$

has dimension greater than 0, i.e., if 1 is an eigenvalue of f^*. In the case where $\dim M = 3$ and f is a pseudo-Anosov homeomorphism of a surface with stretch factor λ, that means that the characteristic polynomial of f^* must be reducible, so that the minimal polynomial of λ has less than maximal degree (see Theorem 8.6.4). Most polynomials with integer coefficients are irreducible, so I am back to expecting that it is exceptional for a 3-manifold to fiber over the circle in several ways unless it happens for topological reasons (for instance when M is the complement of a link with more than one component).

This whole volume is about maps to $\mathbb{R}/\mathbb{Z}$; here it is important to realize that such maps have a cohomological interpretation: $\mathbb{R}/\mathbb{Z}$ is an Eilenberg-MacLane space $K(\mathbb{Z},1)$. The cohomology group $H^1(\mathbb{R}/\mathbb{Z};\mathbb{Z})$ has a natural generator α, represented for instance by dx, where x is the coordinate function on $\mathbb{R}$. Saying that $\mathbb{R}/\mathbb{Z}$ is an Eilenberg-MacLane space $K(\mathbb{Z},1)$ is saying that for any reasonable space[1] X, the map

$$[X, \mathbb{R}/\mathbb{Z}] \to H^1(X; \mathbb{Z}) \quad \text{given by} \quad f \mapsto f^*\alpha \qquad \text{D12.3}$$

is an isomorphism. Here $[X, \mathbb{R}/\mathbb{Z}]$ is the set of homotopy classes of continuous maps $f: X \to \mathbb{R}/\mathbb{Z}$; it is a commutative group because $\mathbb{R}/\mathbb{Z}$ is a commutative group. So it is easy to find lots of maps $X \to \mathbb{R}/\mathbb{Z}$ as soon as $\dim H^1(X; \mathbb{R}) > 1$. Moreover, those maps that are "close" to an original fibration are still fibrations.

Suppose first that a fibration $f : M \to \mathbb{R}/\mathbb{Z}$ is proper, i.e., that M is compact. The circle $\mathbb{R}/\mathbb{Z}$ carries the closed 1-form dx (which on the circle is not d of a function), and f corresponds to the cohomology class of the closed 1-form $\omega := f^*dx$ on M. A sufficiently small neighborhood U of ω in the space of closed 1-forms will consist of nondegenerate 1-forms; in fact the whole cone C over U consists of closed nondegenerate 1-forms. Each element of C represents an element of $H^1(M; \mathbb{R})$ in de Rham cohomology, and a dense subset of C represents elements of $H^1(M; \mathbb{Q})$. Let ω_1 be such a rational closed 1-form; then $k\omega_1$ is integral for some integer k, i.e., $k\omega_1$ represents an element of $H^1(M; \mathbb{Z})$. As such it represents a mapping $f_1 : M \to \mathbb{R}/\mathbb{Z}$, which is a submersion since $k\omega_1$ is nondegenerate. Finally, a proper submersion is a locally trivial fibration (this is a standard result in differential topology; a proof is sketched in Exercise D12.1).

This argument doesn't work if f is not proper: a general submersion is not a fibration. But it does work when M is a 3-manifold that fibers over the circle with fibers that are surfaces of finite type. In that case there is a trick. By Example D5.17 we replace each fiber S_t, $t \in \mathbb{R}/\mathbb{Z}$ by its endpoint compactification $(\overline{S}_t)_E$. By hypothesis the original fibration is locally trivial, so the surfaces $(\overline{S}_t)_E$ are the fibers of a proper fibration $\overline{f} : \overline{M} \to \mathbb{R}/\mathbb{Z}$, to which the argument above can be applied. The added ends $E(S_t)$ form a family of circles transversal to the fibers of f, hence also transversal to the fibers of $\overline{f}_1$ when ω_1 is sufficiently close to ω. Thus they can be removed, providing us with a locally trivial fibration $f_1 : M \to \mathbb{R}/\mathbb{Z}$, again with fibers that are surfaces of finite type.

Exercise D12.1 (Proper submersion is locally trivial fibration) Prove that if X and Y are manifolds and $f : X \to Y$ is a proper submersion, then f is a locally trivial fibration: every point $y \in Y$ has a neighborhood $V \subset Y$ such that there is a diffeomorphism $\varphi : f^{-1}(y) \times V \to f^{-1}(V)$ commuting with the projections to V. *Hint:* The statement is local on Y, so we may assume that $Y = V$ is the open unit cube $(-1, 1)^k$ in some $\mathbb{R}^k$. We will view $X_0 := f^{-1}(\mathbf{0})$ as the "base fiber".

[1] "Reasonable" means, for instance, a space with the homotopy type of a CW ("cell weak") complex, not something like the Hawaiian earring discussed in Appendix A7.2, Volume 1.