Appendix D11
Sullivan's rigidity theorem

Theorem D11.1, first proved by Sullivan [81], is both stronger and weaker than McMullen's rigidity theorem, Theorem 12.3.1. It is stronger because it does not require that the injectivity radius of $\mathbb{H}^3/\Gamma$ be bounded above in its convex core. It is weaker because it requires that the group be finitely generated.

Let Γ be a finitely generated Kleinian group. The Ahlfors finiteness theorem (Theorem 12.2.2) deals with Γ-invariant Beltrami forms carried by the set of discontinuity Ω_Γ. Sullivan's rigidity theorem concerns those carried by the limit set Λ_Γ. This is only interesting if the limit set has positive area. In many cases this is false (all geometrically finite groups with $\Omega_\Gamma \neq \emptyset$, for instance; see Theorem 12.2.7), but in other cases the limit set does have positive area. For example, if the limit set is the entire Riemann sphere $\partial\mathbb{H}^3$, the measure of Λ_Γ is of course nonzero; this case is the main one of interest here.

Theorem D11.1 (Sullivan's rigidity theorem) *If Γ is a finitely generated Kleinian group, the limit set Λ_Γ carries no nontrivial Γ-invariant Beltrami forms.*

The following example, due to Curt McMullen, shows that the hypothesis that Γ be finitely generated is necessary: we describe an infinitely generated group such that there are invariant nontrivial Beltrami forms (lots of them) carried by the limit set.

Example D11.2 We will use the upper halfspace model $\mathbf{H}^3$ of hyperbolic space, bounded by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and parametrized by $(z, t) \in \mathbb{C} \times (0, \infty)$.

Let $i \mapsto D_i$ be a sequence of disjoint discs in $\mathbb{C}$ with radii r_i satisfying $r_i \leq 1$. We will suppose that the union of the D_i is dense in $\mathbb{C}$, but that $\mathbb{C} - \cup D_i$ has positive measure. It is easy to construct such things, essentially the same way one builds Cantor sets in $\mathbb{R}$ of positive measure.

Each convex hull $\widehat{D}_i$ in $\mathbb{H}^3$ is bounded by a hyperbolic plane P_i (a hemisphere in Figure D11.1). Let $\widetilde{\Gamma}$ be the group generated by reflections ρ_i in all the P_i, and let $\Gamma \subset \widetilde{\Gamma}$ be the subgroup of index 2 consisting of orientation-preserving isometries of $\mathbf{H}^3$.

The group $\widetilde{\Gamma}$ is discrete (hence Γ is also): the set $Y := \mathbf{H}^3 - \cup_i \widehat{D}_i$ is a fundamental domain for $\widetilde{\Gamma}$. Indeed, all the reflections of a point $y \in Y$ in one of the P_i have smaller t-coordinate than the t-coordinate of y, and further reflections will just make the t-coordinates smaller yet. A fundamental domain for Γ is given by $Y \cup \rho_i(Y)$.

Set $X := \overline{Y} \cap \overline{\mathbb{C}}$, i.e., the complement of all the open discs D_i. Let us see that X is a subset of the limit set Λ_Γ, hence also of $\Lambda_{\widetilde{\Gamma}}$. Indeed, any $x \in X$ can be approximated by a sequence $D_1, D_2, \dots$; so for any $y \in Y$, the sequence $y, \rho_1(y), \rho_2(\rho_1(y)), \dots$ tends to x; see Figure D11.1.

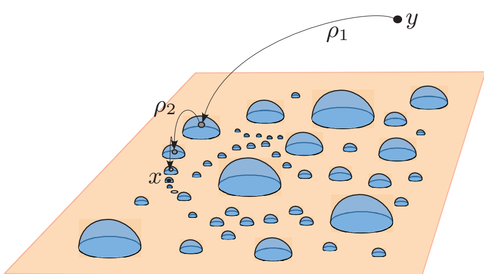

FIGURE D11.1 Every $x \in X$ is in the limit set Λ_Γ: it is an accumulation point of the orbit of y. Each blue dome is the convex hull $\widehat{D}_i$ of the disc D_i.

Thus we have $X \subset \Lambda_\Gamma$ and given any Beltrami form β on X, the Beltrami form

$$\widetilde{\beta} := \begin{cases} \sum_{\gamma\in\Gamma} \gamma^*\beta & \text{on } \bigcup_{\gamma\in\Gamma} \gamma(X) \\ 0 & \text{otherwise} \end{cases} \qquad \text{D11.1}$$

on $\mathbb{P}^1$ is a nontrivial Γ-invariant Beltrami form carried by Λ_Γ. △

The proof of Sullivan's rigidity theorem will require a discussion of *conservative* and *dissipative* actions of groups on probability spaces. Let $(X, \mathcal{A}, \mu)$ be a probability space acted on by a countable group Γ of absolutely continuous transformations[1].

Definition D11.3 (Conservative and dissipative group action) The action of Γ on a measurable subset $Y \subset X$ is *conservative* if for any measurable subset $B \subset Y$ with $\mu(B) > 0$, the set of $\gamma \in \Gamma$ such that $\mu(B \cap \gamma(B)) > 0$ is infinite.

The action of Γ on an invariant measurable subset $Z \subset X$ is *dissipative* if there is a measurable subset $B \subset Z$ such that $\mu(B \cap \gamma(B)) = 0$ for all $\gamma \in \Gamma - \{\mathrm{id}\}$ and

$$\bigcup_{\gamma\in\Gamma} \gamma(B) = Z. \qquad \text{D11.2}$$

[1]i.e., a group of measurable transformations $\gamma : X \to X$ such that $\gamma_*\mu = h\mu$ for some $h \in L^1(X, \mathcal{A}, \mu)$.