Appendix D10
Minimal but not ergodic

Here we see that it is possible for the horizontal foliation of a quadratic differential to be minimal (all leaves dense) but not ergodic. It is difficult to imagine such a measured foliation: the leaves can be colored red and blue, even though red and blue are everywhere, and what you see is purple.

The example was originally due to Veech in the language of interval exchange maps. The treatment below is adapted from Masur and Tabachnikov [56].

Let $T := \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ be the square torus with the quadratic differential induced by dz^2, hence of area 1. Choose two distinct points a and b in T; for convenience we will not put a or b on the "axes" $\mathbb{R}/\mathbb{Z}$ and $i\mathbb{R}/i\mathbb{Z}$; see Figure D10.1, right. Let $\widetilde{a}_0,\ \widetilde{b}_0 \in \mathbb{C}$ be the lifts with real and imaginary parts in $(0, 1)$; see Figure D10.1, left. Use the image $\overline{0} \in T$ of $0 \in \mathbb{C}$ as a base point of $T - \{a, b\}$. Set $\widetilde{I}_0 := [\widetilde{a}_0, \widetilde{b}_0]$ and let I_0 be its image in T.

More generally, for $\mathbf{n} \in \mathbb{Z} + i\mathbb{Z}$, set $\widetilde{b}_{\mathbf{n}} := \widetilde{b}_0 + \mathbf{n}$. Every line segment that joins a and b in T is the image $I_{\mathbf{n}}$ of $\widetilde{I}_{\mathbf{n}} := [\widetilde{a}_0, \widetilde{b}_{\mathbf{n}}]$; see Figure D10.1, right.

By the standard description of covering spaces, the double covers of T ramified at a and b are classified by the elements of

$$\mathrm{Hom}\Big(\pi_1\big(T - \{a, b\}, \overline{0}\big),\ \mathbb{Z}/2\Big) = H^1\Big(T - \{a, b\};\ \mathbb{Z}/2\Big). \qquad \text{D10.1}$$

Remark Note that each double cover comes with a base point $\widehat{0}$ (one of the inverse images of $\overline{0}$ in T): the double covers are defined up to *unique* isomorphism, not just up to isomorphism. △

The space $H^1(T - \{a, b\}; \mathbb{Z}/2)$ is isomorphic to $(\mathbb{Z}/2)^3$; we will use the basis dual to the basis

$$\alpha = \mathbb{R}/\mathbb{Z}, \quad \beta = i\mathbb{R}/i\mathbb{Z}, \quad \gamma \text{ a loop around } a \qquad \text{D10.2}$$

of $H_1(T - \{a, b\}; \mathbb{Z}/2)$. Thus there are eight such covers (corresponding to the eight elements of $(\mathbb{Z}/2)^3$), seven if we require the cover to be connected, and four if we insist that the ramification at a and b be nontrivial; in the latter case the double cover is a surface of genus 2 by the Riemann-Hurwitz formula. These surfaces all come with a quadratic differential q that is the lift of dz^2 on T; it is a holomorphic quadratic differential with two double zeros, in fact $q = \omega^2$, where ω is the lift of dz. For the measure $|q|$ each of these surfaces has area 2.

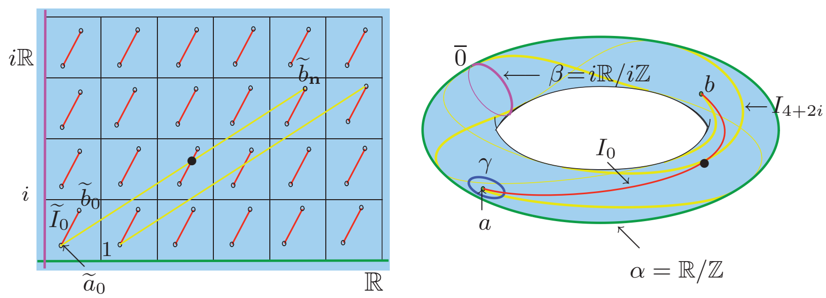

FIGURE D10.1 LEFT: The complex plane $\mathbb{C}$ with the grid $\mathbb{Z}+i\mathbb{Z}$ marked, as well as the points $\widetilde{a}_0$ and $\widetilde{b}_0$, both in $(0,1)\times(0,1)$. The point $\widetilde{b}_{\mathbf{n}}$ corresponds to $\widetilde{b}_0+\mathbf{n}$; here $\mathbf{n} = 4+2i$. The red line segments are the inverse images of the red curve $I_0 \subset T$ at right; they are also the translates of $\widetilde{I}_0$ by elements of $\mathbb{Z}+i\mathbb{Z}$. The yellow lines are translates of $\widetilde{I}_{\mathbf{n}}$. RIGHT: The quotient $T = \mathbb{C}/(\mathbb{Z}+\mathrm{i}\mathbb{Z})$; we see I_0 (the image of $[\widetilde{a}_0, \widetilde{b}_0]$); we also see (yellow) the more complicated I_{4+2i} (the image of $[\widetilde{a}_0, \widetilde{b}_{4+2i}]$). The curves α and β generate the homology of T, and γ is a loop around a. The Riemann surface X on which we will exhibit a non-ergodic minimal measured foliation is the double cover of the torus corresponding to the cohomology class that evaluates to 0 on α and β, and to 1 on γ. We have marked the point where I_0 (red) intersects $I_{\mathbf{n}}$ (yellow); see Exercise D10.1.

The Riemann surface X on which we will exhibit a non-ergodic minimal measured foliation is the double cover corresponding to the cohomology class that evaluates to 0 on α and β, and to 1 on γ. The non-ergodic minimal measured foliation will be the horizontal foliation of the quadratic differential $\left(e^{i\theta}\omega\right)^2$ for an appropriate θ. It will immediately follow that this minimal measured foliation is not ergodic.

I like to visualize X as shown in Figure D10.2, where two copies of the torus are slit along the red curve I_0. (If we had slit along the yellow curve instead, we would get the same surface X, because it is a double cover ramified at a and b, and there only four such double covers; it is straightforward to check which it is. But I find it quite hard to visualize that these two surfaces are the same.)

What happens if we split T_1 and T_2 along $I_{\mathbf{n}}$? The resulting surfaces will be among the four connected double covers with nontrivial ramification at a and b. Which surface we get depends on the parity of the real and imaginary parts of $\mathbf{n}$. The resulting surface is X precisely if the real and imaginary parts of $\mathbf{n}$ are even, since the cohomology class defining the cover evaluates to 0 on β if the real part of $\mathbf{n}$ is even and to 1 if it is odd, and similarly for the imaginary part and α. We will call $I_{\mathbf{n}}$ (and $\widetilde{I}_{\mathbf{n}}$) *admissible* if both real and imaginary parts of $\mathbf{n}$ are even. (This is the case in Figure D10.1, where $\mathbf{n} = 4+2i$.)