Student Solution Manual

Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach, 5th Edition

Notes and errata, complete as of August 28, 2019

This is a list of errata for the first printing of the solution manual.

You have a copy of the first printing if the copyright page contains, after "Printed in the United States of America" the numbers

10 9 8 7 6 5 4 3 2 1

You have a copy of the second printing if the numbers end with 2:

10 9 8 7 6 5 4 3 2

Many thanks to Chester Balestra, Wayne Fincher, Christopher Foo, Radu Grosu, Kabir Kapoor, Alexander Kroeber, Chris Ormandy, Ravi Ramakrishna, Nathaniel Schenker, and Peter Zug for their contributions to this list.

New listings are marked with red.

Page 9 [April 9, 2019] Solution 1.1.9: The first two sentences should read

The number 0 is in the set, since $\operatorname{Re}(w0) = 0$. If a, b are in the set, then $a + b$ is also in the set, since $\operatorname{Re}(w(a + b)) = \operatorname{Re}(wa) + \operatorname{Re}(wb) = 0$. If a is in the set and c is a real number, then ca is in the set, since then $\operatorname{Re}(wca) = c\operatorname{Re}(wa) = 0$.

Page 16 [April 9, 2019] Solution 1.4.23, part c: the third line should be "The limit of $\alpha_{n,n}$ is $\lim_{n\to\infty} \arccos \frac{n}{\sqrt{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}}} = \arccos 0 = \pi/2$. "

Page 18 Solution 1.5.5, part d: In the third line, "the rationals aren't closed" should be "the irrationals aren't closed". In the fourth line, "the irrationals aren't closed" should be "the rationals aren't closed".

Page 18 [May 27, 2017] Exercise 1.5.7 b: "inside the parabola" should be "outside the parabola".

Page 21 Solution 1.5.21: Part b corresponds to an exercise that was dropped from the text, so

a in the text corresponds to a in the solutions.

b in the text has no solution.

c in the textcorresponds to d in the solutions.

d in the text corresponds to e in the solution,

or,
a in the solution manual remains a
c in the solution manual becomes b
d in the solution manual becomes c
e in the solution manual becomes d

Page 22 In the margin note on page 22, "part d of Solution 1.5.21" should be "part c of Solution 1.5.21".

Page 22 Parentheses are missing in the first displayed equation of Solution 1.5.23, part b; moreover, the equation should be labeled (1), not (2):

$$\left|(I+H-I)^{-1}\big((I+H)^2-I^2\big)-2I\right|<\epsilon. \qquad (1)$$

The mistake is repeated in the next equation; the left side should be

$$\left|(I+H-I)^{-1}\big((I+H)^2-I^2\big)-2I\right|$$

Page 27 In the first margin note, "absolute value" should be "length".

Page 29 [Nov. 7, 2017] In Solution 1.9.1, the formula for $D_2f\begin{pmatrix}x\\y\end{pmatrix}$ is missing a 2 in the numerator; the formula should be

$$D_2f\begin{pmatrix}x\\y\end{pmatrix}=\frac{4x^2y^3-2x^4y+2y^5}{(x^2+y^2)^2},$$

with $-2x^4y$ rather than $-x^4y$.

Page 29 [July 9, 2019] Line -2 of the margin note for Solution 1.9.1: "compute", not "compuate".

Page 33 [April 9, 2019] Part b of Solution 1.13: The θ_{y-z} in the next-to-last line should be at the beginning of the last line:

$$\theta_{y-z}=\arcsin\left(\frac{abc}{bc\sqrt{a^2+b^2+c^2}}\right)=\arcsin\left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right).$$

Page 38 [April 9, 2019] We have rewritten part a of Solution 1.37:

a. To get the diagonal entries of A^2 to be 0, we need $a^2=d^2=-bc$. Since $a^2=d^2$ we have $a=\pm d$; we will examine the cases $a=d$ and $a=-d$ separately. If $a=d\neq 0$, we get a contradiction: to get the off-diagonal terms to be 0 we must have either $b=0$ or $c=0$, but then a^2 must be 0, so a and d must be 0. If $a=d=0$, then either $b=0$ or $c=0$. Indeed, the matrices $\begin{bmatrix}0&b\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&0\\c&0\end{bmatrix}$ are matrices whose square is 0.

If $a=-d\neq 0$, then the off-diagonal terms of A^2 are 0. For the diagonal terms to be 0, we must have $bc=-a^2$, so neither b nor c can be 0 and we must have $c=\frac{-a^2}{b}$. Indeed, $A=\begin{bmatrix}a&b\\-\frac{a^2}{b}&-a\end{bmatrix}$ satisfies $A^2=\begin{bmatrix}0&0\\0&0\end{bmatrix}$.

PAGE 38 [April 9, 2019] Solution 1.37, part b, line 3: there should be no period in "either $b = c = 0.$ or $a + d = 0$".

PAGE 38 Solution 1.37, part c: in the next-to-last line, "no such solution" should be "no such solutions".

PAGE 40 [May 14, 2019] Solution 2.1.7: The procedure for the second matrix is incomplete; it should be "multiplying row 2 by 1/2 and subtracting row 3 from row 1, giving $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

PAGE 42 [May 14, 2019] Solution 2.2.7, part a: the value for y is missing a factor of 2 in the denominator; it should be $y = \frac{-b^2-3b+2a}{2(2+a+b)}$.

PAGE 43 [May 14, 2019] Nathaniel Schenker points out that part c of Solution 2.2.11 can be written more simply as a direct computation:

Using $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, compute

$$\begin{aligned}\sum_{k=1}^{n}(2n-1)(n-k+1) &= \sum_{k=1}^{n}(2n-1)(n+1) - \sum_{k=1}^{n}(2n-1)k \\ &= n(2n-1)(n+1) - (2n-1)\frac{n(n+1)}{2} \\ &= n^3 + \frac{n^2}{2} - \frac{n}{2}.\end{aligned}$$

PAGE 44 [May 14, 2019] Solution 2.2.11, part d: the last term for $Q(3)$ should be $-\frac{7}{6}3$, not $\frac{7}{6}9$.

Another way to show that $Q(n) < R(n)$ is to note that for $n \geq 3$,

$$R(n) - Q(n) = \frac{1}{3}n^3 - n^2 + \frac{2}{3}n \geq \frac{1}{3}(3n^2) - n^2 + \frac{2}{3}n = \frac{2}{3}n > 0.$$

PAGE 51 [May 14, 2019] We are replacing Solution 2.4.15 by the following:

2.4.15 Suppose $\vec{\mathbf{w}}$ can be written as a linear combination of the $\vec{\mathbf{v}}_i$ in two ways: $\vec{\mathbf{w}} = \sum_{i=1}^{k} x_i\vec{\mathbf{v}}_i$ and $\vec{\mathbf{w}} = \sum_{i=1}^{k} y_i\vec{\mathbf{v}}_i$, with $x_i \neq y_i$ for at least one i. Then we can write

$$\sum_{i=1}^{k} x_i\vec{\mathbf{v}}_i - \sum_{i=1}^{k} y_i\vec{\mathbf{v}}_i = \vec{\mathbf{0}}, \quad \text{i.e.,} \quad \sum_{i=1}^{k}(x_i - y_i)\vec{\mathbf{v}}_i = \vec{\mathbf{0}},$$

and it follows from Definition 2.4.2 that the $\vec{\mathbf{v}}_i$ are not linearly independent. In the other direction, if any vector can be written as a linear combination of the $\vec{\mathbf{v}}_i$ in at most one way, then $\vec{\mathbf{0}} = \sum_{i=1}^{k} 0\vec{\mathbf{v}}_i = \sum_{i=1}^{k} x_i\vec{\mathbf{v}}_i$ implies that all the x_i are 0, so the $\vec{\mathbf{v}}_i$ are linearly independent.

PAGE 54 [May 14, 2019] We now have two solutions for Exercise 2.5.15:

Solution 1

Since A and B are square, $C = (AB)^{-1}$ is square. We have $CAB = I$ and $ABC = I$, so (using the associativity of matrix multiplication) $(CA)B = I$, so CA is a left inverse of B, and $A(BC) = I$, so BC is a right inverse of A. Since (see the discussion after Corollary 2.2.7) a square matrix with a one-sided inverse (left or right) is invertible, and the one-sided inverse is the inverse, it follows that A and B are invertible.

Solution 2

The second solution reproves the fact that if a square matrix has a left inverse it has a right inverse, and the two inverses are equal.

Note first the following results: if $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations, then

1. the image of T_1 contains the image of $T_1 \circ T_2$, and
2. the kernel of $T_1 \circ T_2$ contains the kernel of T_2.

The first is true because, by the definition of image, for any vector $\vec{\mathbf{v}}$ in $\operatorname{img} T_1 \circ T_2$, there exists a vector $\vec{\mathbf{w}}$ such that $(T_1 \circ T_2)(\vec{\mathbf{w}}) = \vec{\mathbf{v}}$. Since $T_1(T_2(\vec{\mathbf{w}})) = \vec{\mathbf{v}}$, the vector $\vec{\mathbf{v}}$ is also in the image of T_1.

The second is true because for any $\vec{\mathbf{v}} \in \ker T_2$, we have $T_2(\vec{\mathbf{v}}) = \vec{\mathbf{0}}$. Since $T_1(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$, we see that

$$(T_1 \circ T_2)(\vec{\mathbf{v}}) = T_1\big(T_2(\vec{\mathbf{v}})\big) = T_1(\vec{\mathbf{0}}) = \vec{\mathbf{0}},$$

so $\vec{\mathbf{v}}$ is also in the kernel of $T_1 \circ T_2$.

By Proposition 1.2.15, since the matrices A^{-1} and AB are invertible, so is the product

$$B = (A^{-1}A)B = A^{-1}(AB),$$

and $B^{-1} = (AB)^{-1}A$. Note that this uses associativity of matrix multiplication (Proposition 1.2.9).

Recall that the *nullity* of a linear transformation is the dimension of its kernel.

If AB is invertible, then the image of A contains the image of AB by statement 1. So A has rank n, hence nullity 0 by the dimension formula, so A is invertible. Since $B = A^{-1}(AB)$, we have $B^{-1} = (AB)^{-1}A$.

For B, one could argue that $\ker B \subset \ker AB = \{\mathbf{0}\}$, so B has nullity 0, and thus rank n, so B is invertible.

PAGE 56 [May 14, 2019] Solution 2.5.21, b: in lines 2 and 7, parentheses are needed: $T_1(\vec{\mathbf{v}})$ and $T_2(\vec{\mathbf{v}}_i)$.

PAGE 57 [May 14, 2019] Solution 2.6.9, last line of part a: "so they satisfy Definition 2.6.11", not "so they satisfy condition 3 of Definition 2.4.11."

PAGES 59–60 There are errors in Solution 2.7.5, parts a and b. In part a, the correction starts with the second line of the displayed equation introduced by "and finally". Starting with this equation, the solution should be

$$\begin{aligned}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^n &= S \begin{bmatrix} (1+\sqrt{2})^n & 0 \\ 0 & (1-\sqrt{2})^n \end{bmatrix} S^{-1} \\ &= \frac{\sqrt{2}}{4} \begin{bmatrix} (\sqrt{2}-1)(1+\sqrt{2})^n + (\sqrt{2}+1)(1-\sqrt{2})^n & (1+\sqrt{2})^n - (1-\sqrt{2})^n \\ (\sqrt{2}-1)(1+\sqrt{2})^{n+1} + (\sqrt{2}+1)(1-\sqrt{2})^{n+1} & (1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1} \end{bmatrix}.\end{aligned}$$

Multiplying this by $\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we find

$$b_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}$$

The corrected solution to part b is:

b. Since $|1-\sqrt{2}| < 1$, it will contribute practically nothing to b_{1000}, and $\frac{1}{2}(1+\sqrt{2})^{1000}$ has the same number of digits and the same leading digits as b_{1000}. You will find that your calculator will refuse to evaluate this, but using logarithms base 10 for a change, you find

$$\log_{10} b_{1000} \approx 382.475,$$

so b_{1000} has 382 digits, starting with 2983.

PAGE 61 Line 2 of page 61 (expression for c_n towards the end of part c of Solution 2.7.5): there should be no plus sign before λ_1^n.

PAGE 61 [May 14, 2019] Part a of Solution 2.7.7 is not wrong, but we are replacing it with the following:

2.7.7 a. Let A and B be square matrices with λ_A the leading eigenvalue of A and λ_B the leading eigenvalue of B. First let us see that if $A \geq B > \mathbf{0}$, then $\lambda_A \geq \lambda_B$. Choose strictly positive unit eigenvectors $\vec{\mathbf{v}}_A$ and $\vec{\mathbf{v}}_B$ with eigenvalues λ_A and λ_B; then $\lambda_A \geq \lambda_B$:

$$\lambda_A = |\lambda_A \vec{\mathbf{v}}_A| = |A\vec{\mathbf{v}}_A| \overbrace{\geq}^{1} |A\vec{\mathbf{v}}_B| \overbrace{\geq}^{2} |B\vec{\mathbf{v}}_B| = |\lambda_B \vec{\mathbf{v}}_B| = \lambda_B.$$

Inequality (1) follows from the second paragraph of the proof of Theorem 2.7.10. Inequality (2) is true because A and B have only positive entries, $A \geq B$, and $\vec{\mathbf{v}}_B$ has only positive entries.

(This result is of interest in its own right.)

Let A_n be the matrix obtained from A by adding $1/n$ to every entry. Then $A_n > \mathbf{0}$ and by Theorem 2.7.10, A_n has an eigenvector $\mathbf{v}_n \in \overset{\circ}{\Delta}$, where Δ is the set of unit vectors in the positive quadrant Q. Moreover, by the argument above, the corresponding eigenvalues λ_n are nonincreasing and positive, so they have a limit $\lambda \geq 0$.

Since Δ is compact, the sequence $n \mapsto \mathbf{v}_n$ has a convergent subsequence that converges, say to $\mathbf{v} \in \Delta$ (not necessarily in $\overset{\circ}{\Delta}$). Passing to the limit in the equation $A_n\mathbf{v}_n = \lambda_n \mathbf{v}_n$, we see that $A\mathbf{v} = \lambda\mathbf{v}$.

PAGE 61 July 9, 2018] Solution 2.8.3, part b: In the first line, $1/2\sqrt{x} > C$ should be $1/(2\sqrt{x}) > C$.

PAGE 62 Solution 2.8.5, first line: $x_{n+1} = \dfrac{2x_n s + 9}{3x^2}$ should be

$$x_{n+1} = \frac{2x_n^3 + 9}{3x_n^2}.$$

Last line of part b: $\sup_x \in U_1$ should be $\sup\limits_{x \in U_1}$.

PAGE 64 [new, posted August 28, 2019] Solution 2.8.11: The last displayed equation should be

$$\frac{M_1 p(x_1)}{(p'(x_1))^2} \approx \frac{98 \cdot 1.1}{(27)^2} \approx .148 < 1/2.$$

PAGE 66 Solution 2.9.5, fourth line: It should be

"i.e., $2x^2 + x - 1 = 0$", not "i.e., $x^2 + x - 1 = 0$".

PAGE 78 Solution 2.23, line after the first displayed equation: "$n + m \times m$ matrix" should be "$(n + m) \times m$ matrix". We have rewritten the next equation to make the dimensions of the matrix clearer:

Transposing gives the $(n + m) \times m$ matrix

$$\begin{bmatrix} A \\ I \end{bmatrix} H = \begin{bmatrix} AH \\ H \end{bmatrix} = \begin{bmatrix} \widetilde{A} \\ \widetilde{B} \end{bmatrix} = \begin{bmatrix} \begin{matrix} | \\ \vec{\mathbf{a}}_1 \cdots \vec{\mathbf{a}}_k \\ | \end{matrix} & \vec{\mathbf{0}} \\ * & \begin{matrix} | \\ \vec{\mathbf{b}}_1 \cdots \vec{\mathbf{b}}_l \\ | \end{matrix} \end{bmatrix},$$

where $k + l = m$ and by definition k is the index of the last column whose pivotal 1 is in some row whose index is at most n.

PAGE 69 [new, posted August 28, 2019] Exercise 2.10.5, first line:

$$x \le -1/4, \quad \text{not} \quad x \le 1/4.$$

PAGE 70 [new, posted August 28, 2019] Solution 2.10.7: g should be $\mathbf{g}$.

PAGE 84 Solution 2.41: The next-to-last displayed equation is missing a $-n^2 + n$ on the left. It should be

$$2n^3 - 2n^2 + n + 2n^2 - n = 2n^3.$$

ellipsoid1.eps scaled 250 num xxx cap Figure for part b of Solution 3.5.17

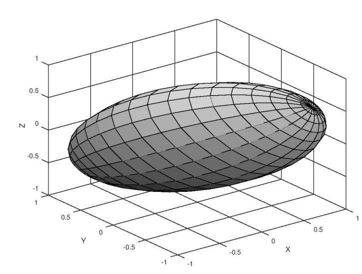

FIGURE FOR SOLUTION 3.5.17.

PAGES 106–107 [July 14, 2017] The solution to Exercise 3.5.17 part b is wrong; the quadratic form was computed incorrectly. The correct solution, courtesy of Chester Balestra, is:

b. The quadratic form corresponding to the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ is

$$2(x^2 + y^2 + z^2 + xy + yz) = 2\left(\left(x + \frac{y}{2}\right)^2 + \left(z + \frac{y}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2\right),$$

which has signature $(3, 0)$. This formula describes an ellipsoid. It is shown in the figure at left, also provided by Chester Balestra.

Page 117 [March 28, 2019] Solution 3.8.5, part b: In the line immediately after the line beginning "Let V be the span of $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ and let W be ... ", the k in $k \le l$ should be j:

for each $j \le l$ set $\vec{\mathbf{u}}'_j = \frac{1}{\sqrt{\mu_j}} A^\top \vec{\mathbf{u}}_j$.

Page 130 [July 9, 2019] Solution 3.27: In the long displayed equation that starts with $\ln(\cos(x_0 + u)) =$, the expression in the first line after the second equal sign:

$$\ln\left(\cos x_0 (1 - \frac{u^2}{2})\right) - u \sin x_0 + o(u^2)$$

is wrong. We have rewritten the solution up to "A similar formula holds for ... ":

Proposition 3.9.11 gives a formula for the mean curvature of a surface in terms of the Taylor polynomial, up to degree 2, of the function of which the surface is a graph. Consider Scherk's surface as the graph of

$$z = f\begin{pmatrix} x \\ y \end{pmatrix} = \ln\left(\frac{\cos x}{\cos y}\right) = \ln(\cos x) - \ln(\cos y). \qquad (1)$$

By Proposition 3.9.10, the Taylor polynomial is computed at the origin, so we make the change of variables $x = x_0 + u$, $y = y_0 + v$. Then the Taylor polynomial, to degree 2, of the first term of equation (1) is

$$\begin{aligned}
\ln(\cos(x_0+u)) &= \ln(\cos x_0 \cos u - \sin x_0 \sin u) \\
&= \ln\left(\cos x_0 \Big(\cos u - \frac{\sin x_0}{\cos x_0} \sin u\Big)\right) \\
&= \ln(\cos x_0) + \ln\Big(\cos u - \frac{\sin x_0}{\cos x_0}\sin u\Big) \\
&\overset{(1)}{=} \ln(\cos x_0) + \ln\left(1 - u\frac{\sin x_0}{\cos x_0} - \frac{u^2}{2} + o(u^2)\right) \\
&\overset{(2)}{=} \ln(\cos x_0) - \left(u\frac{\sin x_0}{\cos x_0} + \frac{u^2}{2}\right) - \frac{1}{2}\left(u\frac{\sin x_0}{\cos x_0} + \frac{u^2}{2}\right)^2 + o(u^2) \\
&= \ln(\cos x_0) - u\frac{\sin x_0}{\cos x_0} - \frac{u^2}{2}\left(1 + \left(\frac{\sin x_0}{\cos x_0}\right)^2\right) + o(u^2).
\end{aligned}$$

To get the equality marked (1) we use

$$\cos u = 1 - \frac{u^2}{2} + o(u^2)$$

and

$$\sin u = u + o(u^2).$$

To get the equality marked (2) we use

$$\ln(1-x) = -x - \frac{x^2}{2} + o(x^2).$$

Page 131 Solution 3.29: In the first displayed equation, $Q_{i,j}^\top A^t op Q_{i,j}$ should be $Q_{i,j}^\top A^\top Q_{i,j}$.

Page 131 Solution 3.29: The last term in the 7th line should be $a_{i,j}(\cos^2\theta - \sin^2\theta)$, not $a_{i,j}(cos^2\theta - \sin^2\theta$.

Page 132 [July 9, 2019] Solution 3.31: In the displayed equation right after "From the first two equations ... ",

$$x_1 = \frac{x_n \dots x_n}{2\lambda} \quad \text{should be} \quad x_1 = \frac{x_2 \cdots x_n}{2\lambda}.$$

and

$$x_2 = \frac{x_1 x_3 \dots x_n}{4\lambda} = \frac{x_2 (x_3 \cdots x_n)^2}{8\lambda}$$

should be

$$x_2 = \frac{x_1 x_3 \cdots x_n}{4\lambda} = \frac{x_2 (x_3 \cdots x_n)^2}{8\lambda^2}$$

In the third line after the displayed equation for λ,

$$x^2 = \frac{x_1}{\sqrt{2}} \quad \text{should be} \quad x_2 = \frac{x_1}{\sqrt{2}}.$$

In the next-to-last line, $nx_1 = 1$ should be $nx_1^2 = 1$, and $x_i = \frac{x_1}{\sqrt{ni}}$ should be $x_i = \frac{1}{\sqrt{ni}}$.

The maximum $\frac{1}{n^{n/2}\sqrt{n!}}$ can also be written $\left(\frac{1}{\sqrt{n}}\right)^n \frac{1}{\sqrt{n!}}$.

PAGE 217 [July 9, 2019] Solution 6.3.15: the vectors $\mathbf{v}$ and $\mathbf{w}$ should not have arrows on them; we are not thinking of column vectors.

PAGE 217 [July 9, 2019] When we reprint the solution manual, we will add this margin note for Solution 6.3.15:

Part b: For $n = 1$, the real change of basis matrix $\widetilde{C}$ that writes $\mathbf{w}$ and $i\mathbf{w}$ in terms of $\mathbf{v}$ and $i\mathbf{v}$ is the 2×2 matrix $\begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix}$, where

$$\begin{aligned} \mathbf{w} &= p_{1,1}\mathbf{v} + p_{2,1}(i\mathbf{v}) \\ i\mathbf{w} &= p_{1,2}\mathbf{v} + p_{2,2}(i\mathbf{v}). \end{aligned}$$

(This uses the notation for the change of basis matrix as defined in Section 2.6.) So by equation (1), the first column of $\widetilde{C}$ is $\begin{bmatrix} a \\ b \end{bmatrix}$ and the second is $\begin{bmatrix} -b \\ a \end{bmatrix}$. Note the similarity of $\widetilde{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ giving clockwise rotation by θ. If we write a complex number z as

$$a + ib = r(\cos\theta + i\sin\theta),$$

then multiplying by z (which is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$) rotates by θ and expands by a factor of r.

PAGE 289 [July 9, 2019] Solution A10.1: The solution given in the first printing of the solution manual is the solution to Exercise A10.1 in the first printing of the text. The second printing of the solution manual will also give the solution to Exercise A10.1 in the second printing of the textbook.

PAGES 290–291 [July 9, 2019] Solution A12.5: We have rewritten the proof of part a in greater detail, to spell out what we mean by "what we know from the chain rule". Here is the new version:

a. Clearly $\operatorname{sgn} y$ is differentiable when $y \neq 0$. The chain rule guarantees that f is continuously differentiable unless $y = 0$ or the quantity under the square root is ≤ 0. But $-x + \sqrt{x^2 + y^2} \geq 0$ and it only vanishes if $y = 0$ and $x \geq 0$.

So to show that f is continuously differentiable on the complement of the half-line $y = 0, x \leq 0$, we need to show that f is continuously differentiable on the open half-line where $y = 0$ and $x > 0$. In a neighborhood of a point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ satisfying $x_0 > 0$ and $y_0 = 0$, we can write (using the Taylor polynomial given in equation 3.4.9, with $m = 1/2$ and $n = 1$),

We can factor out the x to write $-x + \sqrt{x^2 + y^2} = -x + x\sqrt{1 + \frac{y^2}{x^2}}$ because x is positive in a neighborhood of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

$$-x + \sqrt{x^2 + y^2} = -x + x\sqrt{1 + \frac{y^2}{x^2}} = -x + x\left(1 + \frac{1}{2}\frac{y^2}{x^2} + o\left(\frac{y^2}{x^2}\right)\right)$$
$$= \frac{y^2}{2x} + o\left(y^2\right).$$

Since $\sqrt{y^2} = (\operatorname{sgn} y)y$, we see that in a neighborhood of $\begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ with $x_0 > 0$ we have

$$f\begin{pmatrix} x \\ y \end{pmatrix} = (\operatorname{sgn} y)(\operatorname{sgn} y)\left(\frac{y}{2\sqrt{x}} + o(y)\right) = \frac{y}{2\sqrt{x}} + o(y).$$

The function f vanishes identically on the positive x-axis, so we see that f is differentiable on the positive x-axis, with

$$\left[\mathbf{D}f\begin{pmatrix} x \\ 0 \end{pmatrix}\right] = \left[0 \quad \frac{1}{2\sqrt{x}}\right].$$

To see that f is of class C^1 on the positive x-axis, we need to show that the partial derivatives are continuous there. This is a straightforward but messy computation using the chain rule: if $y \neq 0$ we find

$$D_1 f\begin{pmatrix} x \\ y \end{pmatrix} = \operatorname{sgn} y \frac{x - \sqrt{x^2 + y^2}}{4\sqrt{x^2 + y^2}\sqrt{\frac{1}{2}(-x + \sqrt{x^2 + y^2})}}$$

which tends to 0 when y tends to 0 since after cancellations there is a $\sqrt{x - \sqrt{x^2 + y^2}}$ left over. The partial derivative with respect to y is

$$D_2 f\begin{pmatrix} x \\ y \end{pmatrix} = \operatorname{sgn} y \frac{y}{4\sqrt{x^2 + y^2}\sqrt{\frac{1}{2}(-x + \sqrt{x^2 + y^2})}} = \operatorname{sgn} y \frac{y}{4x\frac{(\operatorname{sgn} y)y}{2\sqrt{x}}} + o(y).$$

Again the $\operatorname{sgn}(y)$'s cancel, the y's cancel, one power of 2 cancels, as well as an $\sqrt{x}$, leaving $\frac{1}{2\sqrt{x}}$. So f is continuously differentiable on the open half-line $y = 0$ and $x > 0$, i.e., on the complement of the half-line $y = 0$, $x \leq 0$.