13

Hyperbolization of 3-manifolds that fiber over the circle

One of the great accomplishments of nineteenth century mathematics was showing that all surfaces of genus $g \geq 2$ admit hyperbolic structures. We now present a 3-dimensional version of this theorem: Theorem 13.1.1. This is one of the great accomplishments of twentieth century mathematics.

13.1 Introduction

Let $S := \overline{S} - Z$ be an orientable surface with $\overline{S}$ compact and Z finite; let $f : S \to S$ be an orientation-preserving homeomorphism. Denote by M_f the mapping torus of f, i.e., the quotient of $[0, 1] \times S$ by the equivalence relation that identifies $(0, x)$ to $(1, f(x))$. Every 3-manifold that fibers over the circle is of this form. The map f is called the *holonomy map*; it is defined only up to isotopy and so is really an element of the mapping class group $\mathrm{MCG}(S)$.

Theorem 13.1.1 (Hyperbolization of 3-manifolds that fiber over the circle) *The 3-manifold M_f admits a complete hyperbolic structure if and only if the holonomy map f is homotopic to a pseudo-Anosov homeomorphism.*

Proving this theorem will take the entire chapter. One direction – that if M_f admits a hyperbolic structure, then f is homotopic to a pseudo-Anosov homeomorphism – is easy; at least it follows easily from Theorem 8.1.4 on the classification of homeomorphisms of surfaces. This is the content of Section 13.2.

The proof of the other direction – that if f is homotopic to a pseudo-Anosov map, then M_f carries a hyperbolic structure – is long and hard. We discuss the main idea in Section 13.3. In Section 13.4 we prove the compactness of Bers slices; Sections 13.5–13.8 are devoted to the double limit theorem. We complete the proof of the hyperbolization theorem in Section 13.9.

We begin with Example 13.1.3, which illustrates the theorem for the complement of the figure-eight knot. We will need the notion of *splitting* an n-dimensional manifold-with-boundary X along a properly embedded

$(n-1)$-dimensional submanifold S. We are interested in $n = 3$, splitting a 3-manifold along a properly embedded surface.

Splitting is just "cutting" along S, but cutting is a bit delicate to define, so we adopt a minor variant. We will restrict to the case where S is two sided, i.e., there is a nonvanishing normal vector field on S. In that case there exists an embedding $\varphi : S \times [-1, 1] \to X$ such that $\varphi(x, 0) = x$ and $\varphi(\partial S \times [-1, 1])$ is a subset of ∂X. The image $N(S) := \varphi(S \times [-1, 1])$ is called a *regular neighborhood* of S in X. This neighborhood $N(S)$ has a top, denoted $N(S)^+$, and a bottom, denoted $N(S)^-$:

$$N(S)^+ := \varphi(S \times \{1\}), \quad N(S)^- := \varphi(S \times \{-1\}). \qquad 13.1.1$$

Definition 13.1.2 (Split manifold) The *manifold X split along S* is obtained by choosing an embedding $\varphi : S \times [-1, 1] \to X$ as above and deleting $\varphi(S \times (-1, 1))$. This "split manifold" will be denoted $X^{\wedge S}$.

The procedure is illustrated by Figure 13.1.1.

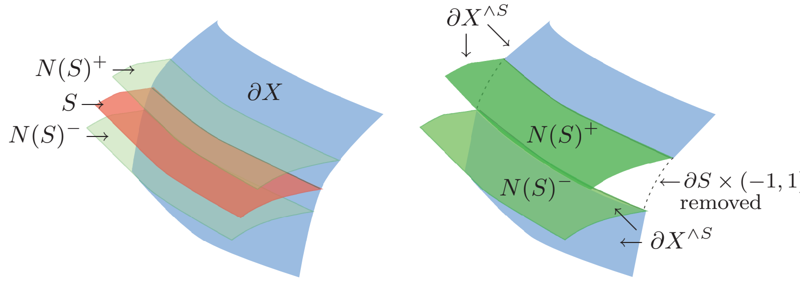

FIGURE 13.1.1 LEFT: The neighborhood $N(S)$ is the region between $N(S)^+$ and $N(S)^-$. RIGHT: The boundary of the "split manifold" $X^{\wedge S} := X - \varphi(S \times (-1, 1))$. The surface S still exists but it is not part of the split manifold, unlike $N(S)^+$ and $N(S)^-$, which are part of $\partial X^{\wedge S}$.

Example 13.1.3 (The complement of the figure-eight knot fibers over the circle) In Example 11.12.13 we showed that the complement M of the figure-eight knot has a complete hyperbolic structure of finite volume. In that example we wrote M as a union of two tetrahedra with faces appropriately identified; here we will give a different decomposition. We use this decomposition to show that M fibers over the circle with fiber a punctured torus T. This makes it possible to compute the holonomy map $f : T \to T$.

A *Seifert surface* is an orientable surface whose boundary is a knot. The "ribbon" S in Figure 13.1.2, left, is a Seifert surface whose boundary is a figure-eight knot K. (It takes quite some work to check that the boundary is a figure-eight knot.) Since the Euler characteristic $\chi(S)$ of S is -1 and ∂S is topologically a circle, S is a torus with a disc removed.

In Figure 13.1.2, right, and in Figure 13.1.3, the knot is thickened into a solid torus T_K. Our manifold M is the complement in S^3 of $\text{int}(T_K)$ (the interior of T_K), so it is a compact manifold-with-boundary.

We will show that S is a fiber of a fibration $M \to \mathbb{R}/\mathbb{Z}$, i.e., that M split along S is homeomorphic to the product $S \times [0, 1]$.

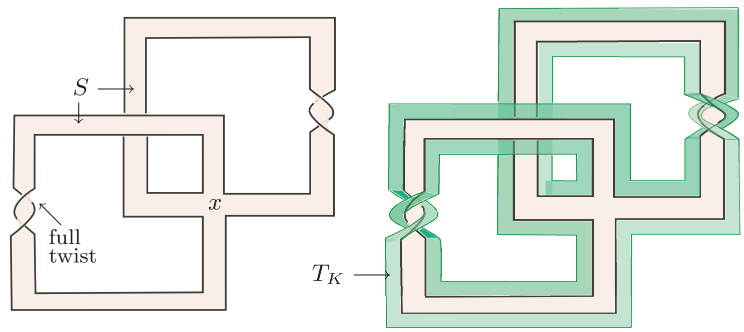

FIGURE 13.1.2 LEFT: A Seifert surface S for the figure-eight knot; S is a surface of genus 1 with one boundary component, the knot. (Although we refer to S as a "ribbon", no ribbon self-intersects as S does at x.) RIGHT: The knot thickened into a solid torus (green). The ribbon is now a properly embedded, incompressible, boundary-incompressible surface S in the complement M of the open torus. The boundary of the ribbon is a curve on the solid torus.

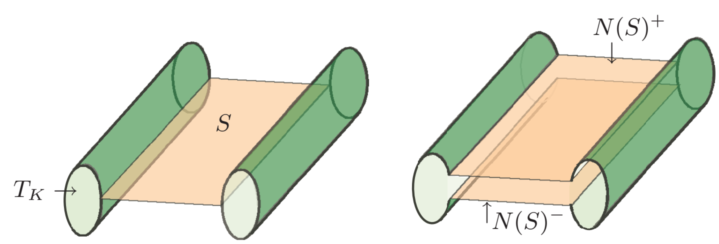

FIGURE 13.1.3 LEFT: A 3-dimensional depiction of part of Figure 13.1.2, right. RIGHT: When M is split along S, the neighborhood $N(S)$ is removed; the top $N(S)^+$ and bottom $N(S)^-$ remain as part of the boundary of M.

The 3-manifold $M^{\wedge S}$ is the complement of a standard handlebody of genus 2, as shown in Figure 13.1.4, left. Hence $M^{\wedge S}$ is also a handlebody of genus 2 in S^3. So we can find two discs D_1 and D_2 in $M^{\wedge S}$; they are drawn as a pink disc bounded by a red circle and a purple disc bounded by a purple circle. Splitting along these discs gives a ball (middle figure); this ball has a front and a back (what $N(S)^+$ and $N(S)^-$ have become after splitting along the discs) separated by a ring formed of four discs and four green bands; the discs are the fronts and the backs of the discs D_1 and D_2.

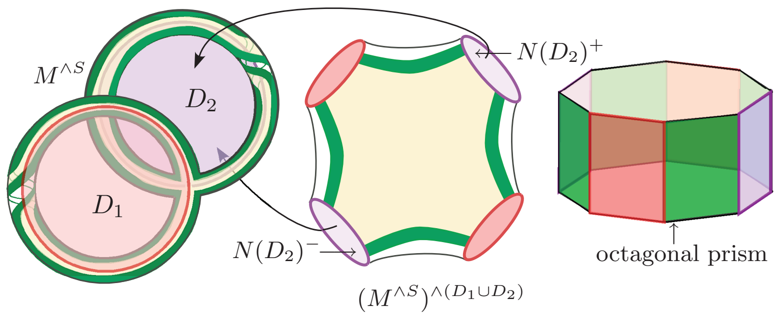

FIGURE 13.1.4 LEFT: If we split the complement M of the figure-eight knot along the surface S, we obtain $M^{\wedge S}$, a handlebody of genus 2 (the complement of the joined rings shown). Two discs D_1, D_2 in $M^{\wedge S}$ are bounded by circles on its boundary (a red circle on the top bounds D_1 and a purple circle on the bottom bounds D_2). MIDDLE: If we split $M^{\wedge S}$ along $D_1 \cup D_2$, we obtain a ball; the torus has become four quadrilaterals (thick green). The two purple discs are the front and back of the purple disc at left; the two pink discs are the front and back of the pink disc. RIGHT: As a polyhedron, this ball is the octagonal prism. The top of the prism corresponds to the yellow region in the middle figure, and the bottom corresponds to the back of the middle figure; the sides are the green quadrilaterals, alternating with the red and purple circles.

The inside and the outside of the ball are homeomorphic. The ball has the polyhedral structure of an octagonal prism (we think of the inside of the prism being the outside of the ball). Think of the prism as having height 1, parametrized by $[0, 1]$; the top and the bottom of the prism are still the two sides of the surface S, and to view them as tori with a disc removed, we must glue the red sides together, and the purple sides together.

Now we can see our fibration: the fibers are the horizontal slices of the octagonal prism. Each fiber has two red edges and two purple edges; these edges are glued together in $M^{\wedge S}$, red to red and purple to purple, so each fiber is a torus with a hole whose boundary is the union of the four green edges, and these boundaries foliate the torus T_K (the boundary of the thickened knot); see Figure 13.1.5.

Thus $M^{\wedge S}$ is homeomorphic to the product $S \times [0, 1]$, and the original manifold M is obtained by gluing $N(S)^+$ to $N(S)^-$ by the holonomy map. Therefore M fibers over the circle $[0, 1]/(0 \sim 1)$, with fibers once-punctured tori; the boundaries of the fibers foliate the torus ∂M.