12

Rigidity theorems

In Chapter 11 we gave an overview of hyperbolic geometry in higher dimensions, especially $n = 3$. In the first half of this chapter we prove three great theorems due to Ahlfors, McMullen, and Mostow. Although the statements look quite different, they are technically closely related: they are all concerned with the construction of Beltrami forms on $\partial\overline{\mathbb{H}}^3$ that are invariant under a Kleinian group Γ. Since these theorems show that such Beltrami forms are very restricted, they are called *rigidity theorems*.

In the second half of the chapter we prove that the central hypothesis of McMullen's rigidity theorem holds for quasi-Fuchsian groups. We need this to prove the hyperbolization theorem for 3-manifolds that fiber over the circle. Along the way we develop several topics, including laminations and pleated surfaces, of great interest in their own right.

12.1 Boundary values of quasi-isometries

In this section we discuss results essential to the rigidity theorems. In Volume 1 (Corollary 4.9.4) we saw that quasiconformal maps $\mathbf{H}^2 \to \mathbf{H}^2$ extend to $\overline{\mathbb{R}}$ as quasisymmetric maps, and that, conversely, any quasisymmetric map $S^1 \to S^1$ extends to a quasiconformal map $\mathbf{D}^2 \to \mathbf{D}^2$. In Chapter 5 we constructed a natural extension, the Douady-Earle extension. In this section we go through the same program one dimension up: extending mappings $\mathbb{H}^3 \to \mathbb{H}^3$ to the sphere at infinity, and conversely extending mappings from the sphere to itself to maps $\mathbb{H}^3 \to \mathbb{H}^3$.

The appropriate class of maps $\mathbb{H}^3 \to \mathbb{H}^3$ to consider is the class of *quasi-isometries*.[1]

Definition 12.1.1 (Quasi-isometric map, quasi-isometry) Let (X, d_X) and (Y, d_Y) be metric spaces, and let C and C' be real numbers. A map $h\colon X \to Y$ is (C, C')-*quasi-isometric* if for all pairs of points $x_1, x_2 \in X$, we have

$$\frac{1}{C} d_X(x_1, x_2) - C' \;\le\; d_Y\Big(h(x_1), h(x_2)\Big) \;\le\; C d_X(x_1, x_2) + C'. \qquad 12.1.1$$

A (C, C')-quasi-isometric map h is a (C, C')-*quasi-isometry* if it is *quasi-surjective*: for all $y \in Y$ there exists $x \in X$ with $d_Y(y, h(x)) \le C'$.

[1] The fact that a quasi-isometric map is not necessarily a quasi-isometry makes me uneasy, but the terminology is standard.

Note that the $+C'$ on the right of inequality 12.1.1 means that a (C, C')-quasi-isometric map need not be continuous. It can "hop" around and has no local regularity whatsoever. The constraint of the inequalities is relevant when points are far apart: two points that are far apart aren't mapped too much closer together or too much further apart.

Note also that a (C, C')-quasi-isometry is an isometry if $C = 1$ and $C' = 0$.

Taking $x_1 = x_2$ shows that we must have $C' \geq 0$ for inequality 12.1.1 to be satisfied. If the diameter of X is infinite, we must also have $C \geq 1$.

Quasi-isometric maps have no regularity; Examples 12.1.2 and Exercise 12.1.5 show that locally they are completely wild.

Example 12.1.2 (Quasi-isometries) Let (Z, d_Z) be a metric space of *finite* diameter D. For any metric space (X, d_X) and any norm $\| \; \|$ on $\mathbb{R}^2$, consider the metric

$$d_{X\times Z}\Big((x_1, z_1), (x_2, z_2)\Big) := \Big\|\Big(d_X(x_1, x_2), d_Z(z_1, z_2)\Big)\Big\| \qquad 12.1.2$$

on $X \times Z$. With this metric, the natural projection $X \times Z \to X$ is a quasi-isometry with $C = 1$ and $C' = D$. In particular, any compact metric space X is quasi-isometric to a point: the projection to a point is a quasi-isometry, and so is the inclusion of a point. Thus the notion of quasi-isometry is really useful only for spaces of infinite diameter. $\triangle$

Exercise 12.1.3 Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b\sin(x)$, with $a \neq 0$, is a (C, C')-quasi-isometry with $C = \sup(|a|, 1/|a|)$ and $C' = 2|b|$, but that the map $\mathbb{C} \to \mathbb{C}$ defined by the same formula $f(z) = az + b\sin(z)$ is not a quasi-isometry. $\diamondsuit$

Exercise 12.1.4 Show that the inclusion of the integers into the reals is a $(1, 0)$-quasi-isometric map, but only a $(1, 1/2)$-quasi-isometry. $\diamondsuit$

Exercise 12.1.5 Let (X, d_X) be a nonempty metric space and choose $\epsilon > 0$. Choose a maximal set $Z \subset X$ of points such that for any two distinct points z_1 and z_2 in Z, we have $d_X(z_1, z_2) \geq \epsilon$.

1. Using Zorn's lemma, show that such a maximal set exists.
2. Show that the inclusion $Z \to X$ is a quasi-isometry.
3. Order the points of Z and define $h : X \to Z$ by mapping every point of X to the nearest point of Z. (If there are several nearest points, choose the point of lowest order.) Show that h is a (C, C')-quasi-isometry for appropriate C, C', which you should compute. $\diamondsuit$

Exercise 12.1.6 Show that quasi-isometry is an equivalence relation on metric spaces. *Hint:* Clearly the identity is a quasi-isometry, and it is easy to see that the composition of two quasi-isometries is a quasi-isometry. It is a lot less clear that if $h: X \to Y$ is a quasi-isometry, then there is a quasi-isometry $g: Y \to X$. One possibility is to define a subset Z of X that is maximal for the property that distinct points are at least $C'(C+1)$ apart. Then show that $h: Z \to h(Z)$ is bijective, and that its inverse is a quasi-isometry. Use Exercise 12.1.5 to finish. $\diamondsuit$

Definition 12.1.7 (Quasi-geodesic, quasi-geodesic ray) Let X be a complete metric space. A (C, C')-*quasi-geodesic* of X is a (C, C')-quasi-isometric map $\gamma: \mathbb{R} \to X$. If we take the domain of γ to be $[0, \infty)$ rather than $\mathbb{R}$, we call γ a (C, C')-*quasi-geodesic ray*.

Theorem 12.1.9 guarantees that in hyperbolic space, a quasi-geodesic stays a bounded distance from a genuine geodesic; it must look like the wide blue curve on the right of Figure 12.1.1. The theorem is related to Theorem 2.3.13 (in Volume 1) on canoeing in the hyperbolic plane; in both cases, the statement is not true in Euclidean space (see the left side of Figure 12.1.1).

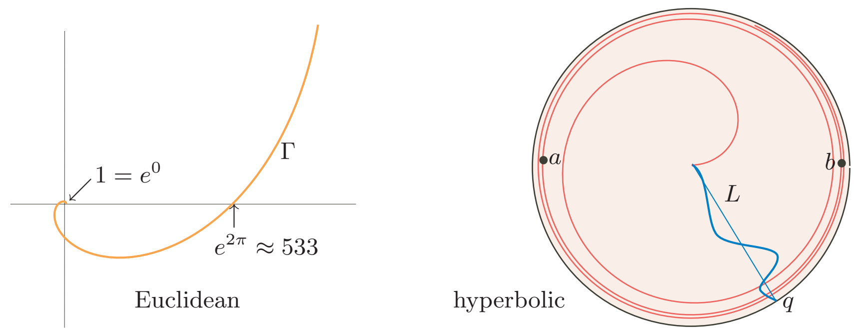
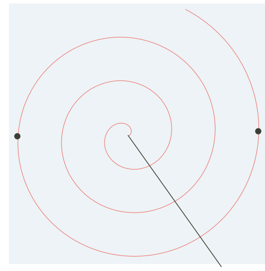


FIGURE 12.1.1 LEFT: The orange curve Γ has equation $r = e^\theta$ in polar coordinates. The successive turns $2k\pi \leq \theta \leq 2(k+1)\pi$ of the spiral are similar. Exercise 12.1.8 asks you to show that if $\gamma : [0, \infty) \to \Gamma$ is parametrization by arc length, then γ is quasi-geodesic. Obviously Γ does not stay a bounded distance from any geodesic ray. RIGHT: By Theorem 12.1.9, a spiral in hyperbolic space is not quasi-geodesic. Indeed, the distance between a and b along the spiral is much greater than the length of the geodesic connecting them. In hyperbolic space, a quasi-geodesic ray must look like the wide blue curve.

In Theorem 12.1.9, the formula for S is not important; what matters is that s, ρ, and especially S depend only on C and C'. The complicated formula is an artifact of the proof. We will denote by $d_{\mathbb{H}}(\mathbf{p}, \mathbf{q})$ the hyperbolic distance between $\mathbf{p}$ and $\mathbf{q}$. To compute the distance between a point and a line, we take the distance to the closest point of the line; to compute the distance between a line and a line, we look at the largest distance between a point of one line to the other line.[2]

Exercise 12.1.8 Show that the curve with polar equation $r = e^\theta$ parametrized by arc length (Figure 12.1.1, left) is a quasi-geodesic. Show that the curve with polar equation $r = \theta$ parametrized by arc length (see the figure at right) is not quasi-geodesic.



Theorem 12.1.9 (Quasi-geodesics and geodesics) *For any $C \ge 1$ and $C' \ge 0$, set*

$$s := \cosh^{-1}\big(2(C+1)^2\big) \tag{12.1.3}$$

$$S := s + (C+1)^2\big(2s + C'(4C+5)\big). \tag{12.1.4}$$

1. *Let $\gamma\colon \mathbb{R} \to \mathbb{H}^n$ be a (C, C')-quasi-geodesic. Then there is a unique geodesic $L \subset \mathbb{H}^n$ such that for all $t \in \mathbb{R}$,*

$$d_{\mathbb{H}}\big(\gamma(t), L\big) \le S. \tag{12.1.5}$$

2. *Let $\gamma\colon [0,\infty) \to \mathbb{H}^n$ be a (C, C')-quasi-geodesic ray. Then $\lim_{t\to\infty} \gamma(t)$ exists in $\partial\overline{\mathbb{H}}^n$, and if $\mathbf{q}$ denotes this limit point, and L the geodesic ray through $\gamma(0)$ and ending at $\mathbf{q}$, then*

$$d_{\mathbb{H}}\big(\gamma(t), L\big) \le S \quad \text{for all } t \in [0,\infty). \tag{12.1.6}$$

The proof below actually works in the much greater generality of Gromov hyperbolic spaces.

Proof of Theorem 12.1.9 The proof requires four results. The first is Exercise 12.1.11; it really uses the fact that we are in hyperbolic geometry. The second is immediate and is the content of Exercise 12.1.12. The third, Lemma 12.1.13, is quite delicate, and contains the main part of the argument. The fourth, Lemma 12.1.14, is a fussy technicality.

Notation 12.1.10 For any line $L \subset \mathbb{H}^n$ and any $s \ge 0$, we denote by $N_s(L) \subset \mathbb{H}^n$ the closed s-neighborhood of L. For $\mathbf{p}, \mathbf{q} \in \overline{\mathbb{H}}^n$, we denote by $[\mathbf{p}, \mathbf{q}]$ the arc of geodesic joining $\mathbf{p}$ to $\mathbf{q}$. For any rectifiable curve $\gamma \subset \mathbb{H}^n$, we denote by $l(\gamma)$ its length.

[2]This is the Hausdorff metric; it may be infinite since the lines are noncompact.