Appendix C4

The mapping class group and outer automorphisms

The mapping class group is an important actor in this book; here we will see that it has an entirely algebraic interpretation, in terms of outer automorphisms. If (X, x) is a *pointed space* (space with a base point), a map $f : X \to X$ is *pointed* if $f(x) = x$. A pointed map induces a homomorphism

$$f_* : \pi_1(X, x) \to \pi_1(X, x) \quad \text{given by} \quad [\gamma] \mapsto [f \circ \gamma]. \qquad C4.1$$

If f is not pointed, there is no such well-defined map; outer automorphisms are a weak substitute for f_*.

For any group G, we denote by $\operatorname{Inn} G \subset \operatorname{Aut} G$ the subgroup of inner automorphisms, i.e., conjugation by elements of G. Clearly $\operatorname{Inn} G$ is a normal subgroup of $\operatorname{Aut} G$ and $\operatorname{Inn} G = G/Z(G)$, where $Z(G)$ denotes the center of G, i.e., the set of elements of G that commute with all elements.

Definition C4.1 (Outer automorphisms) For any group G, the group of *outer automorphisms* of G is $\operatorname{Out} G := \operatorname{Aut} G / \operatorname{Inn} G$.

Note that elements of $\operatorname{Out} G$ are *not* automorphisms of G, unless G is commutative. Example C4.2 shows that $\operatorname{Out} G$ can be tremendously complicated.

Example C4.2 The additive group $\mathbb{Z}^n$ is about the simplest group there is, and $\operatorname{Inn} \mathbb{Z}^n$ is trivial, since $\mathbb{Z}^n$ is commutative. But $\operatorname{Out} \mathbb{Z}^n$ is $\mathrm{GL}_n(\mathbb{Z})$, which is already complicated for $n = 2$ and very mysterious for $n \geq 3$. △

Let (X, x) be a pointed space that is connected, locally connected, and semi-locally simply connected, and let $f : X \to X$ be a continuous map, not necessarily pointed. Any choice of path $\delta : [0, 1] \to X$ with $\delta(0) = x$ and $\delta(1) = f(x)$ induces a map

$$[\delta * f * \delta^{-1}] : \pi_1(X, x) \to \pi_1(X, x), \quad [\gamma] \mapsto [\delta * (f \circ \gamma) * \delta^{-1}], \qquad C4.2$$

where $*$ denotes concatenation. If δ_1 and δ_2 are two such paths, then $[\delta_1 * f * \delta_1^{-1}]$ and $[\delta_2 * f * \delta_2^{-1}]$ are conjugated by $[\delta_2 * \delta_1^{-1}]$. If f is a homeomorphism,

$$[\delta * f * \delta^{-1}] : \pi_1(X, x) \to \pi_1(X, x) \qquad C4.3$$

is an automorphism for any choice of δ, and since any two paths induce conjugate automorphisms, f induces a homomorphism $f_* \in \operatorname{Out}\big(\pi_1(X, x)\big)$.