1.7.15 a. There exists a linear transformation $[\mathbf{D}F(A)]$ such that

$$\lim_{H\to[0]} \frac{\big|F(A+H)-F(A)-[\mathbf{D}F(A)]H\big|}{|H|} = 0.$$

Solution 1.7.15, part (a): The absolute value in the numerator is optional (but not in the denominator: you cannot divide by a matrix).

Since H is an $n\times m$ matrix, the $[0]$ in $\lim_{H\to[0]}$ is the $n\times m$ matrix with all entries 0.

b. The derivative is $[\mathbf{D}F(A)]H = AH^\top + HA^\top$. We found this by looking for linear terms in H of the difference

$$\begin{aligned} F(A+H)-F(A) &= (A+H)(A+H)^\top - AA^\top \\ &= (A+H)(A^\top + H^\top) - AA^\top \\ &= AH^\top + HA^\top + HH^\top; \end{aligned}$$

see remark 1.7.6. The linear terms $AH^\top + HA^\top$ are the derivative. Indeed,

$$\lim_{H\to[0]} \frac{|(A+H)(A+H)^\top - AA^\top - AH^\top - HA^\top|}{|H|}$$
$$= \lim_{H\to[0]} \frac{|HH^\top|}{|H|} \le \lim_{H\to[0]} \frac{|H|\,|H^\top|}{|H|} = \lim_{H\to[0]} |H| = 0.$$

1.7.17 The derivative of the squaring function is given by

$$[\mathbf{D}S(A)]H = AH + HA;$$

substituting $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix}$ gives

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \epsilon & 0 \\ \epsilon & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \epsilon & \epsilon \end{bmatrix} = \begin{bmatrix} \epsilon & 0 \\ 2\epsilon & \epsilon \end{bmatrix}.$$

Computing $(A+H)^2 - A^2$ gives the same result;

$$(A+H)^2 - A^2 = \begin{bmatrix} 1+\epsilon & 2 \\ 2\epsilon & 1+\epsilon \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \epsilon & 0 \\ 2\epsilon & \epsilon \end{bmatrix}.$$

Solution 1.7.17: This may look like a miracle: the expressions should not be equal, they should differ by terms in ϵ^2. The reason why they are exactly equal here is that

$$\begin{bmatrix} 0 & 0 \\ \epsilon & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

1.7.19 Since $\lim_{\vec{\mathbf{h}}\to\mathbf{0}} \frac{|\vec{\mathbf{h}}|\vec{\mathbf{h}}}{|\vec{\mathbf{h}}|} = \mathbf{0}$, the derivative exists at the origin and is the 0 linear transformation, represented by the $n\times n$ matrix with all entries 0.

1.7.21 We will work directly from the definition of the derivative:

$$\begin{aligned} &\det(I+H) - \det(I) - (h_{1,1}+h_{2,2}) \\ &\qquad = (1+h_{1,1})(1+h_{2,2}) - h_{1,2}h_{2,1} - 1 - (h_{1,1}+h_{2,2}) \\ &\qquad = h_{1,1}h_{2,2} - h_{1,2}h_{2,1}. \end{aligned}$$

Each $h_{i,j}$ satisfies $|h_{i,j}| \le |H|$, so we have

$$\frac{|\det(I+H) - \det(I) - (h_{1,1}+h_{2,2})|}{|H|} \le \frac{|h_{1,1}h_{2,2} - h_{1,2}h_{2,1}|}{|H|} \le \frac{2|H|^2}{|H|} = 2|H|.$$

Thus

$$\lim_{H\to 0} \frac{|\det(I+H) - \det(I) - (h_{1,1}+h_{2,2})|}{|H|} \le \lim_{H\to 0} 2|H| = 0.$$