Sample Solutions from the Student Solution Manual

1.2.13 If all the entries are 0, then the matrix is certainly not invertible; if you multiply the 0 matrix by anything, you get the 0 matrix, not the identity. So assume that one entry is not 0. Let us suppose $d \neq 0$. If $ad = bc$, then the first row is a multiple of the second: we can write $a = \frac{b}{d}c$ and $b = \frac{b}{d}d$, so the matrix is $A = \begin{bmatrix} \frac{b}{d}c & \frac{b}{d}d \\ c & d \end{bmatrix}$. (If we had supposed that any other entry was nonzero, the proof would work the same way.) If A is invertible, there exists a matrix $B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. But if the upper left-hand corner of AB is 1, we have $\frac{b}{d}(a'c + c'd) = 1$, so the lower left-hand corner, which is $a'c + c'd$, cannot be 0.

1.3.1 (a) Every linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ is given by a 2×4 matrix, e.g., $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 7 \end{bmatrix}$.

(b) Any row matrix three wide will do, for example, $[1, -1, 2]$; such a matrix takes a vector in $\mathbb{R}^3$ and gives a number.

Remark. On one exam at Cornell University, the first question was "What is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$?" Several students gave answers like "A function taking some vector $\vec{\mathbf{v}} \in \mathbb{R}^n$ and producing some vector $\vec{\mathbf{w}} \in \mathbb{R}^m$."

Besides not being a complete sentence, this is wrong! The student who gave this answer may have been thinking of matrix multiplication. But a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ need not be given by matrix multiplication; for example, consider the mapping $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$. In any case, defining a linear transformation by saying that it is given by matrix multiplication really begs the question, because it does not explain why matrix multiplication is linear.

The correct answer was given by another student in the class:

"A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping $\mathbb{R}^n \to \mathbb{R}^m$ such that for all $\vec{\mathbf{a}}, \vec{\mathbf{b}} \in \mathbb{R}^n$, $T(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = T(\vec{\mathbf{a}}) + T(\vec{\mathbf{b}})$, and for all $\vec{\mathbf{a}} \in \mathbb{R}^n$ and all scalar r, $T(r\vec{\mathbf{a}}) = rT(\vec{\mathbf{a}})$."

Linearity, and the approximation of nonlinear mappings by linear mappings, is a key motif of this book. You must know the definition, which gives you a foolproof way to check whether or not a given mapping is linear. $\triangle$

1.3.7 It is enough to know what T gives when evaluated on the three standard basis vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The matrix of T is $\begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

1.31 Set $f(A) = A^{-1}$ and $g(A) = AA^\top + A^\top A$. Then $F = f \circ g$, and we wish to compute

$$[\mathbf{D}F(A)]H = [\mathbf{D}f \circ g(A)]H = [\mathbf{D}f(g(A))][\mathbf{D}g(A)]H = [\mathbf{D}f(AA^\top + A^\top A)]\ \underbrace{[\mathbf{D}g(A)]H}_{\substack{\text{new increment} \\ \text{for } \mathbf{D}f}}\ . \tag{1}$$

Remember (see Example 1.7.17) that we cannot treat Equation (1) as matrix multiplication. To treat the derivatives of f and g as matrices, we would have to identify $\text{Mat}\,(n, n)$ with $\mathbb{R}^{n^2}$; the derivatives would be $n^2 \times n^2$ matrices. Instead we think of the derivatives as linear transformations.

The linear terms in H of

$$g(A+H)-g(A)=(A+H)(A+H)^\top+(A+H)^\top(A+H)-AA^\top-A^\top A$$

are $AH^\top+HA^\top+A^\top H+H^\top A$; this is $[\mathbf{D}g(A)]H$, which is the new increment for $\mathbf{D}f$.

We know from Proposition 1.7.18 that $[\mathbf{D}f(A)]H=-A^{-1}HA^{-1}$, which we will rewrite as

$$[\mathbf{D}f(B)]K=-B^{-1}KB^{-1} \tag{2}$$

to avoid confusion. We substitute $AH^\top+HA^\top+A^\top H+H^\top A$ for the increment K in Equation (2) and $g(A)=AA^\top+A^\top A$ for B. This gives

$$\begin{aligned}[\mathbf{D}F(A)]H &= [\mathbf{D}f(AA^\top+A^\top A)][\mathbf{D}g(A)]H\\ &= \underbrace{(-AA^\top+A^\top A)^{-1}}_{-B^{-1}}\underbrace{(AH^\top+HA^\top+A^\top H+H^\top A)}_{K}\underbrace{(AA^\top+A^\top A)^{-1}}_{B^{-1}}.\end{aligned}$$

There is no obvious way to simplify this expression.

2.4.7 Let A be an $n\times n$ matrix. The product $A^\top A$ is then

$$\overbrace{\begin{bmatrix}\vdots & \vdots & \dots & \vdots\\ \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_n\\ \vdots & \vdots & \dots & \vdots\end{bmatrix}}^{A}$$

$$\underbrace{\begin{bmatrix}\dots & \vec{\mathbf{a}}_1^\top & \dots\\ \dots & \vec{\mathbf{a}}_2^\top & \dots\\ \dots & \dots & \dots\\ \dots & \vec{\mathbf{a}}_n^\top & \dots\end{bmatrix}}_{A^\top}\underbrace{\begin{bmatrix}|\vec{\mathbf{a}}_1|^2 & \vec{\mathbf{a}}_1\cdot\vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_1\cdot\vec{\mathbf{a}}_n\\ \vec{\mathbf{a}}_2\cdot\vec{\mathbf{a}}_1 & |\vec{\mathbf{a}}_2|^2 & \dots & \vec{\mathbf{a}}_2\cdot\vec{\mathbf{a}}_n\\ \vdots & \vdots & \ddots & \dots\\ \vec{\mathbf{a}}_n\cdot\vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_n\cdot\vec{\mathbf{a}}_2 & \dots & |\vec{\mathbf{a}}_n|^2\end{bmatrix}}_{A^\top A}.$$

The diagonal entries are given by the length squared of the columns of A, since $\vec{\mathbf{a}}_i^\top\vec{\mathbf{a}}_i=\vec{\mathbf{a}}_i\cdot\vec{\mathbf{a}}_i=|\vec{\mathbf{a}}_i|^2$. All other entries are dot products of two different columns of A. If $A^\top A=I$, so that all entries not on the diagonal are 0, while those on the diagonal are 1, then the columns of A are orthogonal and have length 1. Thus they form an orthonormal basis of $\mathbb{R}^n$, and A is said to be orthogonal.

Similarly, if A is orthogonal, then the length of each of its column vectors is 1, so that $A^\top A$ has 1's on the diagonal, and the dot product of two non-identical columns is 0, giving 0's for all other entries of $A^\top A$.

2.5.11 (a) If $ab\neq 2$, then $\dim(\text{Ker }(A))=\mathbf{0}$, so in that case the image has dimension 2. If $ab=2$, the image and the kernel have dimension 1.

(b) This is more complicated. By row operations, we can bring the matrix B to

$$\begin{bmatrix}1 & 2 & a\\ 0 & b & ab-a\\ 0 & 2a-b & a\end{bmatrix}.$$

We now separate the case $b\neq 0$ and $b=0$.

- If $b\neq 0$, then we can do further row operations to bring the matrix to the form

$$\begin{bmatrix}1 & 0 & a-2\frac{ab-a}{b}\\ 0 & 1 & \frac{ab-a}{b}\\ 0 & 0 & -a-(b-2a)\frac{ab-a}{b}\end{bmatrix}.$$ The entry in the 3rd row, 3rd column is $-\dfrac{a}{b}(b^2-2ab+2a)$.

So if $b \neq 0$, and the point $\binom{a}{b}$ is neither on the line $a = 0$ nor on the hyperbola of equation $b^2 - 2ab + 2a = 0$, the matrix has rank 3, whereas if $b \neq 0$ and the point $\binom{a}{b}$ is on one of these curves, the matrix has rank 2.

• If $b = 0$, the matrix is $\begin{bmatrix} 1 & 2 & a \\ 0 & -2a & -a \\ 0 & 0 & a \end{bmatrix}$, which evidently has rank 3 unless $a = 0$, in which case it has rank 1.

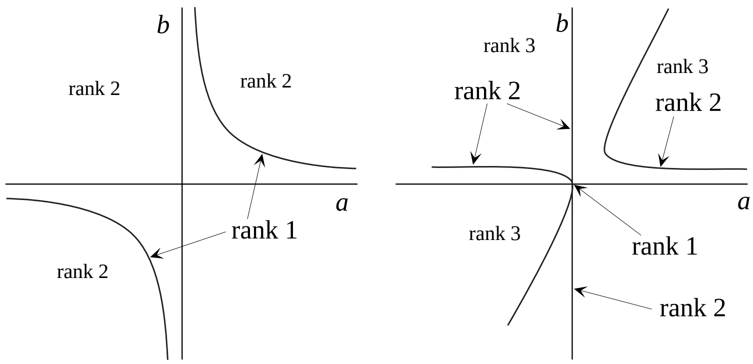

FIGURE FOR SOLUTION 2.5.11. Left: On the curves, the kernel of A has dimension 1 and its image has rank 1. Elsewhere, the rank is 2 (i.e., the kernel has dimension 0). Right: Along the curves and on the b-axis, the image of B has rank 2, i.e., its kernel has dimension 1. At the origin the rank is 1 and the dimension of the kernel is 2. Elsewhere, the kernel has dimension 0 and the rank of the image is 3.

3.1.25 This is a difficult problem; we have criteria guaranteeing that subsets $X \subset \mathbb{R}^n$ are manifolds, but none guaranteeing that they are not. We will outline two possible solutions.

First solution A k-dimensional manifold $X \subset \mathbb{R}^n$ has the property that for any $(k-1)$-dimensional manifold $Y \subset \mathbb{R}^n$ such that $Y \subset X \subset \mathbb{R}^n$, then for any $\mathbf{y} \in Y$ and for any $r > 0$ sufficiently small, Y cuts $X \cap B_r(\mathbf{y})$ into exactly two pieces.

In our case, X will be the set of positions X_2, and Y will be the set of positions where the four vertices are aligned. We must check that Y is a 3-dimensional manifold, but it is pretty clearly parametrized by the position of $\mathbf{x}_1$ and the polar angle of the (straight) linkage.

But if we remove Y from X_2, then $|\mathbf{x}_1 - \mathbf{x}_3| < l_1 + l_2$, so the vertices $\mathbf{x}_2$ and $\mathbf{x}_4$ are not on the line joining $\mathbf{x}_1$ to $\mathbf{x}_3$. Locally $X_2 - Y$ has four pieces: the piece where both $\mathbf{x}_2$ and $\mathbf{x}_3$ are to the left of the line segment going from $\mathbf{x}_1$ to $\mathbf{x}_4$, the piece where they are both to the right, the piece where $\mathbf{x}_2$ is to the left and $\mathbf{x}_4$ to the right, and the piece where $\mathbf{x}_4$ is to the left and $\mathbf{x}_2$ to the right.

Second solution Manifolds are invariant under rotations, so we may assume that our linkage is on the x-axis. If the set of positions is to be a manifold in $\mathbb{R}^8$, then it must locally be the graph of a function expressing four of the variables in terms of the others. We will see that this is impossible for all the possible combinations.

First, we cannot use the positions of any two vertices. Evidently we cannot use any pair of adjacent ones, since the distance between them is fixed, and so they cannot both be chosen freely. We cannot choose either of the pairs of opposite ones either: $\mathbf{x}_1$ and $\mathbf{x}_3$ cannot be further apart than $l_1 + l_2$, and $\mathbf{x}_2$ and $\mathbf{x}_4$ cannot be closer than $|l_1 - l_4|$.

How about the position of one vertex, and either the x or y coordinate of two of the others? First, if we use the position of one, we cannot use the x coordinate of any of the others (why?). But how about $\mathbf{x}_1$ and y_2, y_4? Do these specify a unique position? This is harder to see. Clearly $\mathbf{x}_1, y_2$ and y_4 (with both y's small) specify a unique position of $\mathbf{x}_2$ and $\mathbf{x}_4$ close to the original position. But this leaves two positions of $\mathbf{x}_2$ close to the original. The situation is similar for the other vertices.

Finally, one coordinate for each vertex. We cannot use the x-coordinates of two vertices, because that would allow us to put them closer together or further apart than is allowed. We cannot use all four y-coordinates, as once we have found one position, we can translate it horizontally to find infinitely many positions with the same y-coordinates. Finally, we must investigate the case of one x- coordinate and three y-coordinates. The three y-coordinates determine three of the points, and the fourth must lie on the intersection of two circles, which intersect in two points, both close to the original point.

5.2.3 By Definition 3.1.16, S is not a parametrization, but it is a parametrization by the relaxed definition of a parametrization, Definition 5.2.3. (See the discussion at the beginning of the section).

It is not difficult to show that S is equivalent to the spherical coordinates map using latitude and longitude (Definition 4.10.6). Any point $S(\mathbf{x})$ is on a sphere of radius r, since

$$r^2 \sin^2\varphi \cos^2\theta + r^2 \sin^2\varphi \sin^2\theta + r^2 \cos^2\varphi = r^2.$$

Since $0 \le r \le \infty$, the mapping S is onto.

The angle φ tells what latitude a point is on. Going from 0 to π, it covers every possible latitude. The polar angle θ tells what longitude the point is on; going from 0 to 2π, it covers every possible longitude. However, S is not a parametrization by Definition 3.1.16, because it is not one to one. Trouble occurs in several places. If $r = 0$, $S(\mathbf{x})$ is the origin, regardless of the values of θ and φ. For θ, trouble occurs when $\theta = 0$ and $\theta = 2\pi$; for fixed r and φ, these two values of θ give the same point. For φ, trouble occurs at 0 and π; if $\varphi = \pi$, the point is the south pole of a sphere of radius r, regardless of θ, and if $\varphi = 0$, the point is the north pole of a sphere of radius r, regardless of θ.

By Definition 5.2.3, S is a parametrization. Put simply, the trouble occurs only on a set of 3-dimensional volume 0. In the language of Definition 5.2.3,

$$M = \mathbb{R}^3, \qquad U = \overbrace{[0,\infty)}^{\text{for } r} \times \overbrace{[0,2\pi]}^{\text{for } \theta} \times \overbrace{[0,\pi]}^{\text{for } \varphi},$$

$$X = \Big(\underbrace{\{0\} \times [0,2\pi] \times [0,\pi]}_{\text{trouble when } r=0}\Big) \cup \Big(\underbrace{[0,\infty) \times \{0,2\pi\} \times [0,\pi]}_{\text{trouble when } \theta=0 \text{ or } \theta=2\pi}\Big)$$
$$\cup \Big(\underbrace{[0,\infty] \times [0,2\pi] \times \{0,\pi\}}_{\text{trouble when } \varphi=0 \text{ or } \varphi=\pi}\Big).$$

Thus trouble occurs on a union of five surfaces, which you can think of as five sides of a box whose sixth side is at infinity. The base of the box lies in the plane where $r = 0$; the sides of the base have length π and 2π respectively. The base of the box represents the set labeled "trouble when $r = 0$." Two parallel sides of the box stretching to infinity represent the set labeled "trouble when $\theta = 0$ or $\theta = 2\pi$." The other set of parallel sides represents the set labeled "trouble when $\varphi = 0$ or $\varphi = \pi$." By Proposition 4.3.6 and Definition 5.2.1, these surfaces have 3-dimensional volume 0.

Next let us check (condition 4 of Definition 5.2.3), that the derivative is one to one for all $\mathbf{u}$ in $U - X$. This will be true (Theorem 4.8.5) if and only if the determinant of the derivative is not 0. The determinant is

$$\det\left[\mathbf{D}S\begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix}\right] = \det\begin{bmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{bmatrix} = -2r^2\sin\varphi,$$

which is 0 only if $r = 0$, $\varphi = 0$ or $\varphi = \pi$, which are not in $U - X$.

Next, we will show that $S : (U - X) \to \mathbb{R}^3$ is of class C^1 with locally Lipschitz derivative. We know the first derivatives exist, since we just computed the derivative. They are continuous, since they are all polynomials in r, sine, and cosine, which are continuous (see Theorem 1.5.28 on combining continuous functions). In fact, $S : (U - X) \to \mathbb{R}^3$ is C^∞, since its derivatives of all order are polynomials in r, sine, and cosine, and are thus continuous. We do not need to check anything about Lipschitz conditions because Proposition 2.7.10 says that the derivative of a C^2 function is Lipschitz.

Finally, we need to show that $S(X)$ has 3-dimensional volume 0. If $r = 0$, then $S\begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix}$ is the origin, whatever the values of θ and φ. If $\theta = 0$ we have $S\begin{pmatrix} r \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} r\sin\varphi \\ 0 \\ r\cos\varphi \end{pmatrix}$, which is a surface in the (x, z)-plane. If $\theta = 2\pi$, we get the same surface. If $\varphi = 0$, we get $\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$, the z-axis going from 0 to ∞, and if $\varphi = \pi$ $\begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}$, we get the z-axis going from 0 to $-\infty$.

6.3.7 (a) The basis $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ is direct for V oriented by $dx \wedge dz$ since

$$dx \wedge dz \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right) = \det \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = 2.$$

(b) The basis $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2$ is also direct since

$$dx \wedge dz \left(\begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \right) = \det \begin{bmatrix} 2 & 1 \\ -4 & 5 \end{bmatrix} = 14.$$

Since $\vec{\mathbf{w}}_1 = 2\vec{\mathbf{v}}_1 - 3\vec{\mathbf{v}}_2$ and $\vec{\mathbf{w}}_2 = \vec{\mathbf{v}}_1 + 2\vec{\mathbf{v}}_2$, the change of basis matrix is $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$, with det $+7$.

(c) The determinant is $1/7$, the inverse of the determinant in part (b). (This uses Theorem 4.8.11.) We can also do it by direct computation: since $\vec{\mathbf{v}}_1 = (2/7)\vec{\mathbf{w}}_1 + (3/7)\vec{\mathbf{w}}_2$ and $\vec{\mathbf{v}}_2 = -(1/7)\vec{\mathbf{w}}_1 + (2/7)\vec{\mathbf{w}}_2$, the change of basis matrix is $\begin{bmatrix} 2/7 & -1/7 \\ 3/7 & 2/7 \end{bmatrix}$, with determinant $1/7$. But this is unnecessary work.

6.3.17 (a) The form ω does not orient M. Here are two solutions.

Computational solution:

Parametrize the circle centered at $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ by $\theta \mapsto \begin{pmatrix} 3 + \cos\theta \\ \sin\theta \end{pmatrix}$. A vector in the tangent space to this circle can be written $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, and

$$(x\,dy - y\,dx)\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = (3 + \cos\theta)\cos\theta - \sin\theta(-\sin\theta) = 3\cos\theta + 1.$$

So ω vanishes on the tangent space to the circle at $\theta = \arccos(-1/3)$.

More elegant solution:

As seen in the figure below, at two points on C_2, the tangent spaces to C_2 consist of radial vectors $\begin{bmatrix} x \\ y \end{bmatrix}$. At those points, ω does not orient C_2, since $x\,dy - y\,dx \begin{bmatrix} x \\ y \end{bmatrix} = xy - xy = 0$.

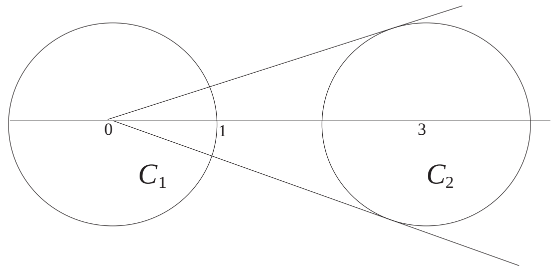

FIGURE FOR SOLUTION 6.3.17, part (a). At two points on C_2, the tangent spaces to C_2 consist of radial vectors $\begin{bmatrix} x \\ y \end{bmatrix}$. The form $\omega = x\,dy - y\,dx$ vanishes on those tangent spaces, so ω does not orient C_2.

6.5.19 (a) No, it does not preserve orientation, since

$$\det \left[\mathbf{D}\gamma \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right] = \det \begin{bmatrix} \cos v \cos w & -u \cos w \sin v & -R \sin w - u \cos v \sin w \\ \cos v \sin w & -u \sin w \sin v & R \cos w + u \cos v \cos w \\ \sin v & u \cos v & 0 \end{bmatrix} = -Ru - u^2 \cos v.$$

This quantity is always negative, because by definition u and R are positive and $Ru > |u^2 \cos v|$ since $-u \le u \cos v \le u$ and $u \le r < R$, so that even when v is between $\pi/2$ and $3\pi/2$ (so that $\cos v$ is negative) still $-Ru - u^2 \cos v < 0$.

(b) The integral becomes

$$\int_0^{2\pi} \int_0^{2\pi} \int_0^r f\left(\gamma \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right) \det \left[\mathbf{D}\gamma \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right] du\,dv\,dw$$

$$= \int_0^{2\pi} \int_0^{2\pi} \int_0^r - \overbrace{(R + u \cos v)^2}^{f(\gamma(\mathbf{u}))} \overbrace{u(R + u \cos v)}^{-\det[\mathbf{D}\gamma]} du\,dv\,dw$$

$$= 2\pi \int_0^{2\pi} \int_0^r (-R^3 u - 3R^2 u^2 \cos v - 3Ru^3 \cos^2 v - u^4 \cos^3 v)\, du\,dv$$

$$= -2\pi \int_0^{2\pi} \left(\frac{R^3 r^2}{2} + R^2 r^3 \cos v + \frac{3Rr^4 \cos^2 v}{4} + \frac{r^5 \cos^3 v}{5} \right) dv$$

$$= -\pi^2 \left(2R^3 r^2 + \frac{3Rr^4}{2} \right)$$

Since the parametrization reverses orientation, we should multiply that integral by -1.

6.7.5 (a) If you approach this is a straightforward manner, integrating everything in sight, this is a long computation. To make it bearable one needs to (1) not compute terms that don't need to be computed, since they will disappear in the limit, and (2) take advantage of cancellations early in the game. To interpret the answer, it also helps to know what one expects to find (more on that later).

Computing the exterior derivative from the definition means computing the integral on the right-hand side of

$$d(z^2\,dx \wedge dy)\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)\big) = \lim_{h \to 0} \frac{1}{h^3} \int_{\partial P_{\mathbf{x}}(h\vec{\mathbf{v}}_1, h\vec{\mathbf{v}}_2, h\vec{\mathbf{v}}_3)} z^2\,dx \wedge dy.$$

Thus we must integrate the 2-form field $z^2\,dx \wedge dy$ over the boundary of the 3-parallelogram spanned by $h\vec{\mathbf{v}}_1, h\vec{\mathbf{v}}_2, h\vec{\mathbf{v}}_3$, i.e., over the six faces of the 3-parallelogram shown below.

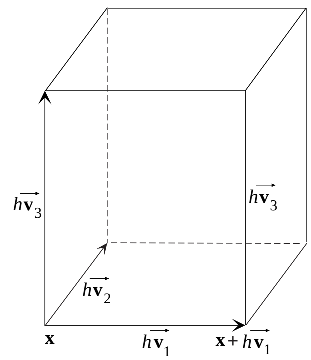

FIGURE FOR SOLUTION 6.7.5. The 3-parallelogram spanned by $h\vec{\mathbf{v}}_1, h\vec{\mathbf{v}}_2, h\vec{\mathbf{v}}_3$.

We parametrize those six faces as follows, where $0 \le s, t \le h$, and $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; we use Proposition 6.6.15 to determine which faces are taken with a plus sign and which are taken with a minus sign:

1. $\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \mathbf{x} + s\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_2$, minus. (This is the base of the box shown above.)
2. $\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \mathbf{x} + h\vec{\mathbf{v}}_3 + s\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_2$, plus. (This is the top of the box.)
3. $\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \mathbf{x} + s\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_3$, plus. (This is the front of the box.)
4. $\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \mathbf{x} + h\vec{\mathbf{v}}_2 + s\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_3$, minus. (This is the back of the box.)
5. $\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \mathbf{x} + s\vec{\mathbf{v}}_2 + t\vec{\mathbf{v}}_3$, minus. (This is the left side of the box.)
6. $\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \mathbf{x} + h\vec{\mathbf{v}}_1 + s\vec{\mathbf{v}}_2 + t\vec{\mathbf{v}}_3$, plus. (This is the right side of the box.)

Note that the two first faces are spanned by $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ (or translates thereof), the base taken with $-$ and the top taken with $+$. The next two faces, with opposite signs, are spanned by $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_3$, and the two sides, with opposite signs, are spanned by $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$.

To integrate our form field over these parametrized domains we use Definition 6.2.1. The computations are tedious, but we do not have to integrate everything in sight. For one thing, common terms that cancel can be ignored; for another, anything that amounts to a term in h^4 can be ignored, since h^4/h^3 will vanish in the limit as $h \to 0$.

We will compute in detail the integrals over the first two faces. The integral over the first face (the base) is the following, where $v_{1,3}$ denotes the third entry of $\vec{\mathbf{v}}_1$ and $v_{2,3}$ denotes the third entry of $\vec{\mathbf{v}}_2$:

$$-\int_0^h \int_0^h \overbrace{(z + sv_{1,3} + tv_{2,3})^2 dx \wedge dy}^{z^2 dx \wedge dy \text{ eval. at } \gamma\begin{pmatrix} s \\ t \end{pmatrix}} \overbrace{(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)}^{\overrightarrow{D_s\gamma}, \overrightarrow{D_t\gamma}} \, ds\, dt$$

$$= -\int_0^h \int_0^h \Big(\underbrace{z^2}_{\text{term in } h^2} + \underbrace{s^2 v_{1,3}^2 + t^2 v_{2,3}^2 + 2st v_{1,3} v_{2,3}}_{\text{terms in } h^4} + \underbrace{2sz v_{1,3} + 2tz v_{2,3}}_{\text{terms in } h^3} \Big) (v_{1,1}v_{2,2} - v_{1,2}v_{2,1}) \, ds\, dt$$

Note that the integral $\int_0^h \int_0^h z^2 \, ds\, dt$ will give a term in h^2, and the next three terms will give a term in h^4; for example, $\int_0^h s^2 v_{1,3}^2 \, ds$ gives an h^3 and $\int_0^h s^2 v_{1,3}^2 \, dt$ gives an h, making h^4 in all. These higher degree terms can be disregarded; we will denote them below by $O(h^4)$. This gives the following integral over the first face:

$$-\int_0^h \int_0^h \big(z^2 + 2szv_{1,3} + 2tzv_{2,3} + O(h^4)\big)(v_{1,1}v_{2,2} - v_{1,2}v_{2,1})\, ds\, dt$$
$$= -\big(h^2z^2 + h^3zv_{1,3} + h^3zv_{2,3} + O(h^4)\big)(v_{1,1}v_{2,2} - v_{1,2}v_{2,1}).$$

Before computing the integral over the second face (the top), notice that it is exactly like the first face except that it also has the term $h\vec{\mathbf{v}}_3$. This face comes with a plus sign, while the first face comes with a minus sign, so when we integrate over the second face, the identical terms will cancel each other. Thus we didn't actually have to compute the integral over the first face at all! In computing the contribution of the first two faces to the integral, we need only concern ourselves with those terms in $(z+hv_{3,3}+sv_{1,3}+tv_{2,3})^2$ that contain $hv_{3,3}$: i.e., $h^2v_{3,1}^2$, $2hsv_{3,3}v_{1,3}$, $2htv_{3,3}v_{2,3}$, and $2zhv_{3,3}$. Integrating the first three would give terms in h^4, so the entire contribution to the integral of the first two faces is

$$\int_0^h \int_0^h \big(2zhv_{3,3} + O(h^4)\big)(v_{1,1}v_{2,2} - v_{1,2}v_{2,1})\, ds\, dt = (2zh^3v_{3,3} + O(h^4))(v_{1,1}v_{2,2} - v_{1,2}v_{2,1}).$$

Similarly, the entire contribution of the second pair of faces is

$$\int_0^h \int_0^h \big(2zhv_{2,3} + O(h^4)\big)(v_{1,1}v_{3,2} - v_{3,1}v_{1,2})\, ds\, dt = (2zh^3v_{2,3} + O(h^4))(v_{1,1}v_{3,2} - v_{3,1}v_{1,2}).$$

(Note that the partial derivatives are different, so we have $(v_{1,1}v_{3,2} - v_{3,1}v_{1,2})$, not $(v_{1,1}v_{2,2} - v_{1,2}v_{2,1})$ as before.) And the contribution of the last pair of faces is

$$\int_0^h \int_0^h \big(2zhv_{1,3} + O(h^4)\big)(v_{2,1}v_{3,2} - v_{2,2}v_{3,1})\, ds\, dt = (2zh^3v_{1,3} + O(h^4))(v_{2,1}v_{3,2} - v_{2,2}v_{3,1}).$$

Dividing by h^3 and taking the limit as $h \to 0$ gives

$$d(z^2\, dx \wedge dy)\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)\big) = (2zv_{3,3})(v_{1,1}v_{2,2} - v_{1,2}v_{2,1}) + (2zv_{2,3})(v_{1,1}v_{3,2} - v_{3,1}v_{1,2})$$
$$+ (2zv_{1,3})(v_{2,1}v_{3,2} - v_{2,2}v_{3,1}).$$

Now it helps to know what one is looking for. Since $d(z^2\, dx \wedge dy)$ is a 3-form on $\mathbb{R}^3$, it is a multiple of the determinant. If you compute $\det[\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3] = \begin{bmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{bmatrix}$, you will see that

$$d(z^2\, dx \wedge dy)\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)\big) = 2z \det[\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3].$$

(Note that in this solution we have reversed our usual rule for subscripts, where the row comes first and column second.)

(b) Using Theorem 6.7.3, we have

$$d(z^2\, dx \wedge dy) \overbrace{=}^{\text{part (e)}} d\, z^2 \wedge dx \wedge dy \overbrace{=}^{\text{part (d)}} (D_1z^2\, dx + D_2z^2\, dy + D_3z^2\, dz) \wedge dx \wedge dy$$
$$= 2z\, dz \wedge dx \wedge dy \overbrace{=}^{\text{Prop. 6.1.21}} 2z\, dx \wedge dy \wedge dz.$$