# Errata and Clarifications: Chapter 5

updated April 14, 2004

**Page 530** Definition 5.1.3: How do we know that $\det(T^\top T) \geq 0$, so that $\sqrt{\det(T^\top T)}$ makes sense? Here is one justification:

Note that

$$(T^\top T)\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = (T^\top T\vec{\mathbf{v}})^\top \vec{\mathbf{v}} = T\vec{\mathbf{v}} \cdot T\vec{\mathbf{v}} > 0.$$

Denote by $A$ the $k \times k$ matrix $T^\top T$ and let $I$ be the $k \times k$ identity matrix, set $0 \leq t \leq 1$, and consider the matrix $\big(tA + (1-t)I$, which we can think of as $A$ (when $t = 1$) being transformed to $I$ (when $t = 0$). Now, for $\vec{\mathbf{v}} \neq \mathbf{0}$, we have

$$\big(tA + (1-t)I\big)\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = t\underbrace{A\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}_{>0} + \underbrace{(1-t)}_{\geq 0}\underbrace{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}_{>0} > 0.$$

This implies that, for $0 \leq t \leq 1$, $\ker\big(tA + (1-t)I = \mathbf{0}$ and thus that $\det\big(tA + (1-t)I$ is never 0 when $0 \leq t \leq 1$. Since when $t = 0$, $\det\big(tA+(1-t)I = 1$, and when $t = 1$, $\det\big(tA+(1-t)I = \det A$, it follows that $\det A > 0$.

**Page 531** The hint for Exercise 5.1.3 is not used in the solution given in the solution manual; in addition, it neglects to define $T$:

$$T = [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k].$$

Here is a solution using the hint:

Set $T = [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]$. Since the vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ are linearly dependent, $\operatorname{rank} T < k$. Further, $\operatorname{Img} T^\top T \subset \operatorname{Img} T^\top$, so

$$\operatorname{rank} T^\top T \leq \operatorname{rank} T^\top \underbrace{=}_{\text{Prop. 2.5.12}} \operatorname{rank} T < k.$$

Since $T^\top T$ is a $k \times k$ matrix with rank $< k$, it is not invertible, hence its determinant is 0, so

$$\operatorname{vol}_k P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \sqrt{\det T^\top T} = 0.$$

**Page 534** Equation 5.2.4: $a_1$ should be $a_i$ in two places, and the "for $a_1, a_2, a_3, \dots$ should be omitted:

$$U = \bigcup_{i=1}^{\infty} \left( a_i - \frac{1}{2^{N+i}},\ a_i + \frac{1}{2^{N+i}} \right). \qquad 5.2.4$$

The next sentence should say "This is an open subset of $\mathbb{R}$ … ," not "This is an open subset of $[0, 1]$ … ."

In Equation 5.2.5, the sum should start at $n = 1$ not $= 1$. On the righthand sides of Equations 5.2.5 and 5.2.6, the denominator should be $2^{N-1}$, not $2^{N-2}$.

**Page 537** Middle margin note: $z$-axis, not $x$-axis, in "you get the equation of the surface obtained by rotating the original curve around the $x$-axis".

**Page 539** In Figure 5.2.4, the top line in the rectangle at right should be darker.

**Page 539** In Theorem 5.2.10 we used the word *diffeomorphism* without defining it. A diffeomorphism is a differentiable mapping with differentiable inverse.

**Page 541** Three lines after Equation 5.3.2: "sum them," not "summ them."

**Page 541** Definition 5.3.1: This definition is not wrong, but it is unfortunate that we restricted ourselves to this special case instead of defining the integral of a function over a manifold. In subsequent editions, we will replace this definition by something like

**Definition 5.3.1 (Integral with respect to volume, over a manifold).** Let $M \subset \mathbb{R}^n$ be a smooth $k$-dimensional manifold, $U$ a pavable subset of $\mathbb{R}^k$, and $\gamma : U \to M$ a parametrization according to Definition 5.2.3. Let $f : M \to \mathbb{R}$ be a function. Then $f$ is integrable over $M$ with respect to volume if the last integral below exists, and then the integral is

$$\int_M f(\mathbf{x})|d^k\mathbf{x}| = \int_{\gamma(U)} f(\mathbf{x})|d^k\mathbf{x}| = \int_U f\big(\gamma(\mathbf{u})\big)\Big(|d^k\mathbf{x}|\,\big(P_{\gamma(\mathbf{u})}\big(\overrightarrow{D_1}\gamma(\mathbf{u}),\dots,\overrightarrow{D_k}\gamma(\mathbf{u})\big)\big)\Big)|d^k\mathbf{u}|$$
$$= \int_U f\big(\gamma(\mathbf{u})\big)\sqrt{\det([\mathbf{D}\gamma(\mathbf{u})]^\top[\mathbf{D}\gamma(\mathbf{u})])}\;|d^k\mathbf{u}|. \qquad 5.3.3$$

Such an integral is sometimes referred to as the integral of a density, as opposed to the integral of a differential form.

If $f = 1$, the integral above gives the volume of $M$.

A corresponding change would then need to be made to Proposition 5.3.2 and its proof.

In several examples and exercises we actually use the above definition of "integral of a function with respect to volume."

**Page 545** Line 2, plural, not singular: "the intersection of the surfaces of equations".

Equation 5.3.26: the second line should end with $d\theta$.

Equation 5.3.27: This equation should not have a $d\theta$ at the end. It should have a period.

**Page 549** In three places, $D_2 f$ should be $D_3 f$: the last line of Equation 5.3.45 should be

$$1 + (D_1 f)^2 + (D_2 f)^2 + (D_3 f)^2;$$

In the second line of Equation 5.3.45, three closing parentheses aren't opened. The line should be

$$= \det \begin{bmatrix} 1+(D_1 f)^2 & (D_1 f)(D_2 f) & (D_1 f)(D_3 f) \\ (D_1 f)(D_2 f) & 1+(D_2 f)^2 & (D_2 f)(D_3 f) \\ (D_1 f)(D_3 f) & (D_2 f)(D_3 f) & 1+(D_3 f)^2 \end{bmatrix}$$

Equation 5.3.46 should be

$$\int_U \sqrt{1 + (D_1 f)^2 + (D_2 f)^2 + (D_3 f)^2}|d^3\mathbf{x}|,$$

and the left-hand side of the first line of Equation 5.3.48 should be

$$\int_{B_{\mathbf{0}}(R)} \sqrt{1 + (D_1 f)^2 + (D_2 f)^2 + (D_3 f)^2}|d^3\mathbf{x}|.$$

**Page 550** The caption to Table 5.3.3 would perhaps be clearer as follows:

Computing the volume of the $n$-dimensional unit ball in $\mathbb{R}^n$, for $n = 1, \dots, 5$, and for the $n$-dimensional unit sphere in $\mathbb{R}^{n+1}$, for $n = 0, 1 \dots, 5$. (The 0-dimensional sphere in $\mathbb{R}$ consists of the two points $-1$ and 1.)

**Page 551** Exercise 5.3.2: "Use the result of Exercise 5.3.1 (a)", not "use Equation 5.3.1 … ".

**Page 552** first margin note: the earth's circumference, not diameter!

Exercise 5.3.12: The total curvature of a curve $C$ is $\int_C \kappa \, |d^1\mathbf{x}|$.

**Page 556** Exercise 5.6: Some subscripts got forgotten, and one superscript is wrong. It should be:

(a) Show that $w'_{n+1}(r) = v_n(r)$.

(b) Show that $v_n(r) = r^n v_n(1)$.

(c) Derive Equation 5.3.49, using $w_{n+1}(1) = \displaystyle\int_0^1 w'_{n+1}(r)\, dr$.