Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach

Notes and Errata for the Third Edition

Complete as of June 3, 2009

We omit from this list minor notational inconsistencies that should cause no confusion (for example, sometimes we write $\mathrm{vol}_n A$ and sometimes $\mathrm{vol}_n(A)$, with parentheses).

Many thanks to

Travis Allison, David Besser, Tara Holm, Thomas Madden, Richard Palas, Lewis Robinson, Leonard Smiley, Leo Trottier, and Charles Yu

for their contributions to this list

Mathematical errors

PAGE 18 Third line from bottom: $k > j+1$ should be $k > j-1$ and $k < j+1$ should be $k < j-1$. In the next two lines: In the subscripts, $j+1$ should be $j-1$.

PAGE 19 First line: the subscript $j+2$ should be $j-2$, and $j+3$ should be $j-3$.

PAGE 36 In figure 1.1.3, the point labeled $\mathbf{a}$ should be $\mathbf{b}$ and the point labeled $\mathbf{b}$ should be $\mathbf{a}$. The existing figure shows $\overrightarrow{\mathbf{b}-\mathbf{a}}$ not $\overrightarrow{\mathbf{a}-\mathbf{b}}$.

PAGE 41 The vectors given for exercise 1.1.5 should be replaced, for instance by the vectors in $\mathbb{R}^n$

$$\begin{bmatrix}1\\1\\\vdots\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\\\vdots\\n-1\\n\end{bmatrix}, \quad \begin{bmatrix}0\\0\\3\\4\\\vdots\\n-1\\n\end{bmatrix}$$

PAGE 98 In the 5th line, the first $\mathbf{x}$ should be $\mathbf{x}_n$: "there exists N such that when $n > N$ we have $|\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{x}_0)| < \epsilon$."

PAGE 108 Caption for figure 1.6.2: If the fractional part of a number α is between 0 and 1/2, then $\sin 2\pi\alpha \geq 0$; if it is between 1/2 and 1, then $\sin 2\pi\alpha \leq 0$. (We are changing from strict inequalities because "between 0 and 1/2" is ambiguous.)

PAGE 117 2 lines before equation 1.6.26, replace "$|u| = \rho < 1$" by "$|u| = \rho \leq 1$". Line after equation 1.6.26, replace "$|b_j u^j| \geq$" by "$|b_j u^j| >$, in two places.

PAGE 129 Proof of proposition 1.7.11, next to last line: "$\lim_{\vec{\mathbf{h}}\to\mathbf{0}}(\mathbf{f}(\mathbf{a}+\vec{\mathbf{h}})-\mathbf{f}(\mathbf{a}))$ must also be $\mathbf{0}$."

PAGE 147 Exercise 1.8.10: The beginning of part a should read

"Show that if a function $f:\mathbb{R}^2\to\mathbb{R}$ can be written $\varphi(x^2+y^2)$ for some differentiable function $\varphi:\mathbb{R}\to\mathbb{R}$, then ... "

PAGE 158 We gave a wrong value for e. It is $e=2.71828\ldots$.

PAGE 159 Exercise 1.34 contains two errors. First, "such that $h(\mathbf{x})=-h(\mathbf{x})$." should have been "such that $h(-\mathbf{x})=-h(\mathbf{x})$." Second, it is not true that under the given hypotheses, "f has directional derivatives in all directions, at all points of $\mathbb{R}^2$." This exercise should be replaced by the following:

1.34 Let $h:\mathbb{R}\to\mathbb{R}$ be a C^1 function, periodic of period 2π, and define $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = rh(\theta).$$

a. Show that f is a continuous real-valued function on $\mathbb{R}^2$.

b. Show that f is differentiable on $\mathbb{R}^2-\{\mathbf{0}\}$.

c. Show that all directional derivatives of f exist at $\mathbf{0}$ if and only if $h(\theta)=-h(\theta+\pi)$ for all θ.

d. Show that f is differentiable at $\mathbf{0}$ if and only if $h(\theta)=a\cos\theta+b\sin\theta$ for some numbers a and b.

PAGE 171 On the left side of equation 2.2.9, x_i, not x_1.

PAGE 185 Definition 2.4.5 of linear independence does not allow for the case $k=n$. It should be

Definition 2.4.5 (Linear independence). The vectors $\vec{\mathbf{v}}_1,\ldots,\vec{\mathbf{v}}_k$ in $\mathbb{R}^n$ are *linearly independent* if, when a vector $\vec{\mathbf{w}}\in\mathbb{R}^n$ can be written as a linear combination of $\vec{\mathbf{v}}_1,\ldots,\vec{\mathbf{v}}_k$, it can be done so in only one way:

$$\sum_{i=1}^k x_i\vec{\mathbf{v}}_i = \sum_{i=1}^k y_i\vec{\mathbf{v}}_i \quad\text{implies}\quad x_1=y_1,\ x_2=y_2,\ldots,x_k=y_k.$$

The vectors $\vec{\mathbf{e}}_1,\vec{\mathbf{e}}_2\in\mathbb{R}^3$ are linearly independent: there are vectors in $\mathbb{R}^3$ that cannot be written as a linear combination of $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$, but if a vector in $\mathbb{R}^3$ can be written as a linear combination of those two vectors, there is only only one way to do it.

PAGE 240 The last margin note should be deleted, and the second equation in example 2.8.8 should be $D_2(D_1f)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = D_2\underbrace{(2+y^3)}_{D_1f} = 3y^2$. We could define f using a,b,c rather than x,y,z, but once we have defined it using x,y,z, the partial derivatives need to be computed using those variables. Then we could *evaluate* the partial derivatives at $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to get $3b^2$.

PAGE 275 The augmented matrix in the margin should be

$$\Big[[\mathbf{D_p F(c)}] \big| - [\mathbf{D_{np} F(c)}] \Big].$$

PAGE 277 Exercise 2.10.16: The same letter A is used to denote both a specific matrix and a general matrix A in the neighborhood U. This should be:

2.10.16 The matrix $A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ satisfies $A_0^3 = I$.

True or false? There exists a neighborhood $U \subset \operatorname{Mat}(3,3)$ of I and a continuously differentiable function $g : U \to \operatorname{Mat}(3,3)$ with $g(I) = A_0$ and $(g(A))^3 = A$ for all $A \in U$ (i.e., $g(A)$ is a cube root of A).

PAGE 303 In exercise 3.1.4, the half-axes should be

$$\mathbb{R}^+_z, \quad \mathbb{R}^+_x, \quad \mathbb{R}^+_y, \quad \mathbb{R}^-_z, \quad \mathbb{R}^-_x, \quad \mathbb{R}^-_y$$

PAGE 305 Exercise 3.1.23: We should have mentioned that this exercise concerns example 3.1.8. The points should have been bold: $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$.

PAGE 307 In equations 3.2.1 and 3.2.2, $[\mathbf{Df}(\mathbf{c})]$ should be $[\mathbf{Df}(\mathbf{b})]$.

PAGE 313 Exercise 3.2.11, part d: “if A is invertible” should be “if A is orthogonal”.

PAGE 347 Equation 3.6.8: $D^i D^j$ should be $D_i D_j$.

PAGE 403–404 There is an error in the proof of proposition 4.1.23. Here is a corrected proof:

By proposition 4.1.19, this is true if A is a parallelogram, in particular if A is a cube $C \in \mathcal{D}_N$. Assume A is a subset of $\mathbb{R}^n$ whose volume is well defined. This means that $\mathbf{1}_A$ is integrable, or, equivalently, that

$$\lim_{N\to\infty} \sum_{C \in \mathcal{D}_N, C \subset A} \operatorname{vol}_n(C) = \lim_{N\to\infty} \sum_{C \in \mathcal{D}_N, C \cap A \neq \emptyset} \operatorname{vol}_n(C),$$

and that the common limit is $\operatorname{vol}_n(A)$.

Since

$$\bigcup_{C \in \mathcal{D}_N, C \subset A} tC \ \subset \ tA \ \subset \bigcup_{C \in \mathcal{D}_N, C \cap A \neq \emptyset} tC,$$

and since $\mathrm{vol}_n(tC) = t^n \,\mathrm{vol}_n(C)$ for every cube, this gives

$$\begin{aligned} t^n \sum_{C \in \mathcal{D}_N, C \subset A} \mathrm{vol}_n(C) &= \int \sum_{C \in \mathcal{D}_N, C \subset A} \mathbf{1}_{tC} \le L(\mathbf{1}_{tA}) \le U(\mathbf{1}_{tA}) \le \int \sum_{C \in \mathcal{D}_N, C \cap A \neq \emptyset} \mathbf{1}_{tC} \\ &= t^n \sum_{C \in \mathcal{D}_N, C \cap A \neq \emptyset} \mathrm{vol}_n(C). \end{aligned}$$

Since the outer terms have a common limit as $N \to \infty$, it follows that $L(\mathbf{1}_{tA}) = U(\mathbf{1}_{tA})$, and that

$$\mathrm{vol}_n(tA) = \int \mathbf{1}_{tA}(\mathbf{x}) |d^n \mathbf{x}| = t^n \,\mathrm{vol}_n(A). \qquad \square$$

PAGE 429 The last two sentences in the first paragraph of example 4.4.3 should be "The sum of the lengths of these intervals ... is $\epsilon\,(1/2 + 1/4 + 1/8 + \dots) = \epsilon$, and the length of the union is $< \epsilon$, since some of the intervals overlap."

PAGE 473 For proposition 4.8.22 to be true, we must allow the matrix P to be *unitary*, the complex generalization of orthogonal. This requires complex inner product, which we have not defined. They aren't really that much more difficult than real inner products, but there is a complex conjugate which shows up and makes all the computations more complicated; in our experience students find them a lot more difficult to deal with. So for pedagogical reasons, we omitted them. We have replaced the statement as follows:

Proposition 4.8.22. *If A is an $n \times n$ complex matrix, then there exists an invertible matrix P such that $P^{-1}AP$ is upper triangular. Equivalently, there is a basis $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$ such that the matrix of A in that basis is upper triangular.*

PAGE 483 Exercise 4.9.7, part b: $T \mapsto \mathrm{vol}_n(T(Q))$, not $Q \mapsto \mathrm{vol}_n(T(Q))$.

PAGE 523 In the second margin note, "(r, φ, θ)-space" should be "(r, θ, φ)-space" and "(φ, θ)-plane" should be "the (θ, φ)-plane".

PAGE 562 Third line of definition 6.1.23: $\mathbf{x} \in U$, not $\mathbf{x} \in \mathbb{R}^n$.

PAGE 543 Equation 5.4.5: $K(\mathbf{x})$ should be $|K(\mathbf{x})|$.

PAGE 581 Solution 6.3.15: There is some confusion, when speaking of change of basis matrices, about "from" and "to". In the exercise we asked for the change of basis matrix going from the $\mathbf{v}_i$ to the $\mathbf{w}_i$, but what is actually written expresses the basis vectors $\mathbf{w}_i$ in terms of the $\mathbf{v}_j$: the columns of the change of basis matrix C are the coordinates of the $\mathbf{w}_i$ with respect to the $\mathbf{v}_j$.

PAGE 585 In proposition 6.4.7, "with U connected" should be "with $U - X$ connected, where X is as in definition 5.2.3." With the relaxed definition of parametrization 5.2.3, U might be connected, but $U - X$ might not. In that case checking at a single point will only give "orientation preserving" in the set of points that can be connected to that point in $U - X$.

PAGE 608 Equations 6.6.6 and 6.6.7: $\vec{\mathbf{v}}_k$, not $\vec{\mathbf{v}}_n$

PAGE 620 Remark 6.7.3: At the end of the first paragraph, some subscripts are wrong. This should be

"the integral over a face should be something of the form $h^k(a_0 + a_1 h + a_2 h^2 \dots)$; it would appear that dividing by h^{k+1} would give a term $a_0/h \dots$ ".

PAGE 627 Second paragraph of the margin note: "if φ is a 0-form, then $d\varphi$ is a 1-form", *not* " … then $d\varphi$ is a 2-form"

PAGE 637 Line before equation 6.9.21: "steady current", not "current"

PAGE 638 There are two errors in the right side of equation 6.9.25: the grad should be $-$ grad and the exponent in the denominator should be 1, not 2:

$$\frac{\overrightarrow{\mathbf{x}-\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^3} = -\operatorname{grad} \frac{1}{|\mathbf{x}-\mathbf{y}|},$$

PAGE 643 Equation 6.9.54 is wrong. It should be

$$f(t,\mathbf{x}) = g(\mathbf{x}\cdot\mathbf{v} - at)$$

PAGE 646 Exercise 6.9.6: In keeping with the correction page 643, the equation should be

$$f(t,\mathbf{x}) = g(\mathbf{x}\cdot\mathbf{v} - at)$$

PAGE 670 Third line of exercise 6.15: subset of S^3, not subset of M.

PAGE 675 In equation A1.3, the inf should be over $l \le -k$, not $l \ge k$. Or, one could keep $l \ge k$ and change the other l's in the formula to $-l$'s.

PAGE 687 In proposition A5.1, we had defined $\mathbf{f}$ on the closed set $\overline{U}$, so differentiability doesn't really make sense. The proposition should read:

Proposition A5.1. *Let $U \subset \mathbb{R}^n$ be an open ball, $V \subset \mathbb{R}^n$ a neighborhood of $\overline{U}$, and $\mathbf{f} : V \to \mathbb{R}^m$ a differentiable mapping whose derivative satisfies the Lipschitz condition*

$$\big|[\mathbf{Df}(\mathbf{x})] - [\mathbf{Df}(\mathbf{y})]\big| \le M|\mathbf{x}-\mathbf{y}| \text{ for all } \mathbf{x},\mathbf{y} \in \overline{U}. \tag{A5.10}$$

Then for $\mathbf{x},\mathbf{y} \in \overline{U}$,

$$\Big|\underbrace{\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})}_{\text{increment to } \mathbf{f}} - \underbrace{[\mathbf{Df}(\mathbf{x})](\mathbf{y}-\mathbf{x})}_{\substack{\text{linear approx.} \\ \text{of increment to } \mathbf{f}}}\Big| \le \frac{M}{2}|\mathbf{y}-\mathbf{x}|^2. \tag{A5.11}$$

PAGE 691 First margin note: $\vec{\mathbf{h}}_0$ should be $|\vec{\mathbf{h}}_0|$ and $2\vec{\mathbf{h}}_0$ should be $2|\vec{\mathbf{h}}_0|$.

PAGE 692 Exercise A5.1: In the second displayed equation, $(\alpha - 1)$ should be $(\alpha + 1)$.

PAGE 695 Equation A7.8 should have absolute values:

$$|\vec{\mathbf{r}}(\vec{\mathbf{k}})| = |\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}}) - \mathbf{g}(\mathbf{y}_0)| \le 2|L^{-1}|\,|\mathbf{y}_0 + \vec{\mathbf{k}} - \mathbf{y}_0)|.$$

There are more serious problems with the proof of the inverse function theorem. Below is a new version.

We will begin by proving that $\mathbf{f}$ is injective on W_0, which will prove that $\mathbf{g}$ is unique. To show that $\mathbf{f}$ is injective on W_0, we will show that the function

$$\mathbf{F}(\mathbf{z}) \stackrel{\text{def}}{=} \frac{1}{2R|L^{-1}|} L^{-1}\Big(\mathbf{f}\big(\mathbf{x}_0 + 2R|L^{-1}|\mathbf{z}\big) - \mathbf{y}_0\Big)$$

satisfies the hypotheses of lemma A1.

The function $\mathbf{F}$ is designed so that it is defined in the unit ball of $\mathbb{R}^n$, with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and $[\mathbf{DF}(\mathbf{0})] = I$. Let us check that the Lipschitz condition on $\mathbf{f}$ (see equation 2.10.12) translates into the simpler Lipschitz condition of lemma A1:

$$\begin{aligned} |[\mathbf{DF}(\mathbf{z}_1)] - [\mathbf{DF}(\mathbf{z}_2)]| &= \Big|L^{-1}\Big[\mathbf{Df}\Big(\mathbf{x}_0 + 2R|L^{-1}|\mathbf{z}_1\Big)\Big] - L^{-1}\Big[\mathbf{Df}\Big(\mathbf{x}_0 + 2R|L^{-1}|\mathbf{z}_2\Big)\Big]\Big| \\ &\le |L^{-1}|\frac{1}{2R|L^{-1}|^2}\Big|2R|L^{-1}|\mathbf{z}_1 - 2R|L^{-1}|\mathbf{z}_2\Big| = |\mathbf{z}_1 - \mathbf{z}_2|. \end{aligned}$$

Lemma A1. *Let B be the unit ball of $\mathbb{R}^n$, and let $\mathbf{F} : B \to \mathbb{R}^n$ be a C^1 mapping such that*

$$\mathbf{F}(\mathbf{0}) = \mathbf{0}, \quad [\mathbf{DF}(\mathbf{0})] = I, \quad \text{and} \quad \big|[\mathbf{DF}(\mathbf{x})] - [\mathbf{DF}(\mathbf{y})]\big| \le |\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in B$. Then $\mathbf{F}$ is injective.

Proof. Using corollary 1.9.2, we can write

$$\begin{aligned} |\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})| &= \big|(\mathbf{x} - \mathbf{y}) + \big(\mathbf{F}(\mathbf{x}) - \mathbf{x}) - (\mathbf{F}(\mathbf{y}) - \mathbf{y})\big)\big| \\ &= \big|(\mathbf{x} - \mathbf{y}) + \big(\mathbf{F} - I)(\mathbf{x}) - (\mathbf{F} - I)(\mathbf{y})\big| \\ &\ge |\mathbf{x} - \mathbf{y}| - \sup_{\mathbf{z} \in [\mathbf{x},\mathbf{y}]} \Big|[\mathbf{DF}(\mathbf{z})] - I\Big|\,|\mathbf{x} - \mathbf{y}| \\ &= |\mathbf{x} - \mathbf{y}| - \sup_{\mathbf{z} \in [\mathbf{x},\mathbf{y}]} \Big|[\mathbf{DF}(\mathbf{z})] - [\mathbf{DF}(\mathbf{0})]\Big|\,|\mathbf{x} - \mathbf{y}| \\ &\ge |\mathbf{x} - \mathbf{y}| - \sup_{\mathbf{z} \in [\mathbf{x},\mathbf{y}]} \Big|\mathbf{z} - \mathbf{0}\Big|\,|\mathbf{x} - \mathbf{y}| \\ &= |\mathbf{x} - \mathbf{y}|\big(1 - \sup(|\mathbf{x}|, |\mathbf{y}|)\big). \end{aligned}$$

(To go from the next-to-last line to the last line, we use that $\sup_{\mathbf{z} \in [\mathbf{x},\mathbf{y}]}\big|\mathbf{z}\big| = \sup(|\mathbf{x}|, |\mathbf{y}|)$, since the point of a line segment farthest from the origin is always one of the endpoints.) Thus $\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$. $\square$

It follows that $\mathbf{g}$ is the unique map $V \to W_0$ such that $\mathbf{f} \circ \mathbf{g}$ is the identity on V. If $\mathbf{g}_1$ is such a map, then $\mathbf{g}_1(\mathbf{y})$ is an inverse image of $\mathbf{y}$ under $\mathbf{f}$ that is an element of W_0, and there is at most one (hence exactly one) such inverse image, namely $\mathbf{g}(\mathbf{y})$.

Proving that g is continuous on V.

The inequality in the proof of lemma A1 gives us a bit more, it tells us that $\mathbf{g}$ is continuous, and even gives a modulus of continuity. Indeed, for any $\mathbf{x}_1, \mathbf{x}_2 \in W_0$ we have

$$\begin{aligned}|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)| &= 2R|L^{-1}| \left| L\left(\mathbf{F}\left(\frac{\mathbf{x}_1 - \mathbf{x}_0}{2R|L^{-1}|}\right) - \mathbf{F}\left(\frac{\mathbf{x}_2 - \mathbf{x}_0}{2R|L^{-1}|}\right)\right)\right| \\ &\geq 2R\left|\mathbf{F}\left(\frac{\mathbf{x}_1 - \mathbf{x}_0}{2R|L^{-1}|}\right) - \mathbf{F}\left(\frac{\mathbf{x}_2 - \mathbf{x}_0}{2R|L^{-1}|}\right)\right| \\ &\geq 2R\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{2R|L^{-1}|}\left(1 - \frac{\sup(|\mathbf{x}_1 - \mathbf{x}_0|, |\mathbf{x}_2 - \mathbf{x}_0|)}{2R|L^{-1}|}\right) \\ &= \frac{1}{|L^{-1}|}\left(1 - \frac{\sup(|\mathbf{x}_1 - \mathbf{x}_0|, |\mathbf{x}_2 - \mathbf{x}_0|)}{2R|L^{-1}|}\right)|\mathbf{x}_1 - \mathbf{x}_2|.\end{aligned}$$

Thus if $\mathbf{x}_1 = \mathbf{g}(\mathbf{y}_1)$ and $\mathbf{x}_2 = \mathbf{g}(\mathbf{y}_2)$, with

$$\sup(|\mathbf{x}_1 - \mathbf{x}_0|, |\mathbf{x}_2 - \mathbf{x}_0|) \leq 2R'|L^{-1}|$$

for some $R' < R$, then

$$|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)| \geq |\mathbf{x}_1 - \mathbf{x}_2|\frac{R - R'}{|L^{-1}|R}, \quad \text{i.e.,} \quad |\mathbf{g}(\mathbf{y}_1) - \mathbf{g}(\mathbf{y}_2)| \leq \frac{R|L^{-1}|}{R - R'}|\mathbf{y}_1 - \mathbf{y}_2|. \qquad (A1)$$

Changing the base point

Note that to show the existence of $R > 0$ and of $\mathbf{g}$, the only hypotheses about $\mathbf{f}$ we used were that $\mathbf{f}(\mathbf{x}_0) = \mathbf{y}_0$, that $[\mathbf{Df}(\mathbf{x}_0)]$ is invertible, and that $\mathbf{x} \mapsto [\mathbf{Df}(\mathbf{x})]$ is Lipschitz in a neighborhood of $\mathbf{x}_0$. For any points $\mathbf{y}'_0 \in V$ and $\mathbf{x}'_0 = \mathbf{g}(\mathbf{y}'_0)$, the same hypotheses are true.

Write $\mathbf{g} = \mathbf{g}_{\mathbf{x}_0, \mathbf{y}_0}$. For any $\mathbf{y}'_0 \in V$ and $\mathbf{x}'_0 \stackrel{\text{def}}{=} \mathbf{g}(\mathbf{y}'_0)$, there is an analogous map $\mathbf{g}_{\mathbf{x}'_0, \mathbf{y}'_0}$. This map also specifies an inverse image of $\mathbf{y}$ under $\mathbf{f}$, and since $\mathbf{f}$ is injective on W_0, it must be the same inverse image in some neighborhood of $\mathbf{y}'_o$. Thus if we prove that $\mathbf{g} = \mathbf{g}_{\mathbf{x}_0, \mathbf{y}_0}$ is differentiable at $\mathbf{y}_0$, the same proof will show that $g_{\mathbf{x}'_0, \mathbf{y}'_0}$ is differentiable at $\mathbf{y}'_0$, hence $\mathbf{g}$ is also differentiable at $\mathbf{y}'_0$, since it coincides with $g_{\mathbf{x}'_0, \mathbf{y}'_0}$ in a neighborhood of $\mathbf{y}'_0$.

Let us see that $\mathbf{g}(V)$ is open. From the argument above, if $\mathbf{g}(V)$ contains a neighborhood of $\mathbf{x}_0$, then it will contain a neighborhood of ll $\mathbf{x}'_0 \in \mathbf{g}(V)$. By the injectivity of $\mathbf{f}$ on W_0, $\mathbf{g}(V)$ does contain a neighborhood of $\mathbf{x}_0$: if $\mathbf{x}$ is sufficiently close to $\mathbf{x}_0$ then $\mathbf{f}(\mathbf{x})$ is in V so we can consider the element $\mathbf{x}_1 \stackrel{\text{def}}{=} \mathbf{g}(\mathbf{f}(\mathbf{x})$ of W_0, and

$$\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{f}(\mathbf{x})$$

which by injectivity of $\mathbf{f}$ implies $\mathbf{x} = \mathbf{x}_1 \in \mathbf{g}(V)$.

Proving that g is differentiable at $\mathbf{y}_0$

Here we show that $\mathbf{g}$ is differentiable at $\mathbf{y}_0$, with derivative $[\mathbf{Dg}(\mathbf{y}_0)] = L^{-1}$, i.e., that

$$\lim_{\vec{\mathbf{k}} \to \vec{\mathbf{0}}} \frac{(\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}}) - \mathbf{g}(\mathbf{y}_0)) - L^{-1}\vec{\mathbf{k}}}{|\vec{\mathbf{k}}|} = \vec{\mathbf{0}}. \qquad (A2)$$

When $|\mathbf{y}_0 + \vec{\mathbf{k}}| \in V$, define $\vec{\mathbf{r}}(\vec{\mathbf{k}})$ to be the increment to $\mathbf{x}_0$ that under $\mathbf{f}$ gives the increment $\vec{\mathbf{k}}$ to $\mathbf{y}_0$:

$$\mathbf{f}\big(\mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}})\big) = \mathbf{y}_0 + \vec{\mathbf{k}}, \tag{A3}$$

or, equivalently,

$$\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}}) = \mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}}). \tag{A4}$$

Substitute the right side of equation (A4) for $\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}})$ in the left side of equation (A2), remembering that $\mathbf{g}(\mathbf{y}_0) = \mathbf{x}_0$. This gives find

$$\lim_{\vec{\mathbf{k}}\to\vec{\mathbf{0}}} \frac{\mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}}) - \mathbf{x}_0 - L^{-1}\vec{\mathbf{k}}}{|\vec{\mathbf{k}}|} = \lim_{\vec{\mathbf{k}}\to\vec{\mathbf{0}}} \frac{\vec{\mathbf{r}}(\vec{\mathbf{k}}) - L^{-1}\vec{\mathbf{k}}}{|\vec{\mathbf{k}}|} \frac{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}$$

$$= \lim_{\vec{\mathbf{k}}\to\vec{\mathbf{0}}} \frac{L^{-1}\left(L\vec{\mathbf{r}}(\vec{\mathbf{k}}) - \overbrace{\Big(\mathbf{f}\big(\mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}})\big) - \mathbf{f}(\mathbf{x}_0)\Big)}^{\vec{\mathbf{k}}\text{ by equation (A3)}}\right)}{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|} \frac{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}{|\vec{\mathbf{k}}|}.$$

Since $\mathbf{f}$ is differentiable at $\mathbf{x}_0$, the term

$$\frac{L\vec{\mathbf{r}}(\vec{\mathbf{k}}) - \mathbf{f}(\mathbf{x}_0 + \vec{\mathbf{r}}(\vec{\mathbf{k}})) + \mathbf{f}(\mathbf{x}_0)}{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}$$

has limit $\vec{\mathbf{0}}$ as $\vec{\mathbf{r}}(\vec{\mathbf{k}}) \to \vec{\mathbf{0}}$. The differentiability of $\mathbf{g}$ at $\mathbf{y}_0$ (equation A7.3) will follow from part 5 of theorem 1.5.23 if we show that $\vec{\mathbf{r}}(\vec{\mathbf{k}}) \to \vec{\mathbf{0}}$ when $\vec{\mathbf{k}} \to \vec{\mathbf{0}}$, fast enough that $|\vec{\mathbf{r}}(\vec{\mathbf{k}})|/|\vec{\mathbf{k}}|$ remains bounded. Equation (A4) tells us that

$$|\vec{\mathbf{r}}(\vec{\mathbf{k}})| = |\mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}}) - \mathbf{g}(\mathbf{y}_0)|$$

and since $\mathbf{g}$ is continuous, $|\vec{\mathbf{r}}(\vec{\mathbf{k}})|$ does tend to 0 with $|\vec{\mathbf{k}}|$. The proof that $\mathbf{g}$ is continuous (equation (A1)) shows more. Let $\mathbf{x} = \mathbf{g}(\mathbf{y}_0 + \vec{\mathbf{k}})$; as $\vec{\mathbf{k}}$ tends to $\vec{\mathbf{0}}$, the length $|\mathbf{x} - \mathbf{x}_0|$ also tends to 0 since $\mathbf{g}$ is continuous, in particular it will satisfy $|\mathbf{x} - \mathbf{x}_0| < RL^{-1}$ when $|\vec{\mathbf{k}}|$ is sufficiently small. Equation (A1) says that for these values of $\vec{\mathbf{k}}$ we have

$$\frac{|\vec{\mathbf{r}}(\vec{\mathbf{k}})|}{|\vec{\mathbf{k}}|} \le \frac{2R|L^{-1}|}{R - R/2} = 4|L^{-1}|$$

so $|\vec{\mathbf{r}}(\vec{\mathbf{k}})|/|\vec{\mathbf{k}}|$ remains bounded as $\vec{\mathbf{k}} \to \vec{\mathbf{0}}$.

Proving that g is continuously differentiable on V. This follows immediately from the formula $[\mathbf{Dg}(\mathbf{y})] = [\mathbf{Df}(\mathbf{g}(\mathbf{y}))]^{-1}$, derived from the chain rule.

Proving equation 2.10.14 Suppose $|\mathbf{x} - \mathbf{x}_0| < R_1$. Then

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| \le |\mathbf{x} - \mathbf{x}_0| \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big| < R_1 \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big|. \tag{A5}$$

So if $R_1 \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big| < R$, then $\mathbf{f}(\mathbf{x})$ is in V. We find a bound for $|[\mathbf{Df}(\mathbf{z})]|$, when $|\mathbf{z} - \mathbf{x}_0| < R_1$:

$$|[\mathbf{Df}(\mathbf{z})] - [\mathbf{Df}(\mathbf{x}_0)]| = |[\mathbf{Df}(\mathbf{z})] - L| \underbrace{\le}_{\text{Eq. 2.10.12}} \frac{1}{2R|L^{-1}|^2}\,|\mathbf{z} - \mathbf{x}_0| < \frac{R_1}{2R|L^{-1}|^2}$$

so

$$|[\mathbf{Df}(\mathbf{z})]| \leq |L| + \frac{R_1}{2R|L^{-1}|^2}, \text{ i.e., } \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big| \leq |L| + \frac{R_1}{2R|L^{-1}|^2}.$$

Substituting this bound for $R_1 \sup_{|\mathbf{z}-\mathbf{x}_0|<R_1} \big|[\mathbf{Df}(\mathbf{z})]\big|$ in equation (A5), we see that if

$$R \geq \left(|L| + \frac{R_1}{2R|L^{-1}|^2}\right) R_1, \qquad (A6)$$

then $|\mathbf{x}-\mathbf{x}_0| < R_1$ implies $\mathbf{f}(\mathbf{x}) \in V$. Then $\mathbf{g}(\mathbf{f}(\mathbf{x}))$ is an inverse image of $\mathbf{f}(\mathbf{x})$ in W_0, but since $\mathbf{f}$ is injective on W_0 there only is one, so $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$ and $\mathbf{x} \in \mathbf{g}(V)$.

We leave it to the reader to check that the largest R_1 satisfying equation (A6) is

$$R_1 = R|L^{-1}|^2 \left(-|L| + \sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}}\right). \qquad A7.15$$

Page 733 Since proposition 4.8.22 has been changed, the proof needs to be changed:

Proof. The proof is by induction on n. It is obvious if $n = 1$, so suppose $n \geq 2$ and assume the result for all $(n-1) \times (n-1)$ matrices.

Find an eigenvector $\vec{\mathbf{v}}_1$ with eigenvalue λ_1 (which exists by the fundamental theorem of algebra and the procedure described in section 2.7). Choose vectors $\vec{\mathbf{w}}_2, \dots, \vec{\mathbf{w}}_n$ such that $\vec{\mathbf{v}}_1, \vec{\mathbf{w}}_2, \dots, \vec{\mathbf{w}}_n$ is a basis of $\mathbb{R}^n$. Then $T \stackrel{\text{def}}{=} [\vec{\mathbf{v}}_1, \vec{\mathbf{w}}_2, \dots, \vec{\mathbf{w}}_n]$ is invertible, and since $\vec{\mathbf{v}}_1$ is an eigenvector of A, the first column of $B \stackrel{\text{def}}{=} T^{-1}AT$ is $\lambda_1\vec{\mathbf{e}}_1$, i.e., we can write

$$B \stackrel{\text{def}}{=} T^{-1}AT = \left[\begin{array}{c|c} \lambda_1 & \beta \\ \hline 0 & \\ \vdots & \widetilde{B} \\ 0 & \end{array}\right].$$

where β is some $1 \times (n-1)$ matrix, and $\widetilde{B}$ is an $(n-1) \times (n-1)$ matrix.

By our inductive hypothesis, we can find an invertible matrix $\widetilde{Q}$ such that $\widetilde{Q}^{-1}\widetilde{B}\widetilde{Q}$ is upper triangular. Set

$$Q = \left[\begin{array}{c|c} 1 & \dots \quad 0 \quad \dots \\ \hline 0 & \\ \vdots & \widetilde{B} \\ 0 & \end{array}\right] \quad \text{so that} \quad Q^{-1}BQ = \left[\begin{array}{c|c} \lambda_1 & \beta\widetilde{Q} \\ \hline 0 & \\ \vdots & \widetilde{Q}^{-1}\widetilde{B}\widetilde{Q} \\ 0 & \end{array}\right]. \qquad A18.17$$

In particular, $Q^{-1}BQ$ is upper triangular. Set $P = TQ$, then

$$P^1AP = Q^{-1}T^{-1}ATQ = Q^{-1}BQ \qquad A18.18$$

is upper triangular. $\square$

Page 734 Corollary A18.1 does not follow from the new version of proposition 4.8.22.

Pages 738–740 Because we no longer can use corollary A18.1, some changes are required in the subsection “completing the proof of the change of variables formula”. At the bottom of page 738, “we will denote by Z the union of these cubes” should become “We will denote by Z the union of the closure of these cubes”.

The top of page 739 (up to Lemma A19.6) should become

We will also require that N_1 be large enough so that $Z \subset U$. Then $X \cup Z$ is a compact subset of U. Call M the Lipschitz ratio of $[\mathbf{D}\Phi]$, and set

$$K = \sup_{\mathbf{x} \in X \cup Z} |\det[\mathbf{D}\Phi(\mathbf{x})]| \quad \text{and} \quad L = \sup_{\mathbf{y} \in Y} |f(\mathbf{y})|. \qquad A19.25$$

We know that K is well defined because $|\det[\mathbf{D}\Phi(\mathbf{x})]|$ is continuous on $X \cup Z$.

In the remainder of the proof, K^n should be replaced by K.

Non-mathematical errors, very minor errors, and miscellaneous notes

PAGE 3 First margin note: "n terms", not "nt terms"

PAGE 14 Example 0.4.8: In the second paragraph, 1st and 2nd lines, replace "positive" by "nonnegative".

PAGE 17 Note that real numbers are actually bi-infinite decimals with 0's to the left: a number like 3.0000 . . . is actually

$$\ldots 00003.0000 \ldots.$$

By convention, leading 0's are usually omitted (one exception being credit card expiration dates: to an issuing bank, March is 03, not 3).

PAGE 18 6th line from bottom: "since $[x]_j$ is one of them", not "since $[a]_j$ is one of them". 3rd line from bottom: $j-1$ not $j+1$, in two places. Next 2 lines: b_{j-1}, not b_{j+1}, in three places.

PAGE 19 First line: b_{j-2} and b_{j-3}, not b_{j+2} and b_{j+3}.

PAGE 19 In example 0.5.6, 2.020202 should be 2.020202 . . . :

For example, $2.020202\ldots = 2 + 2(.01) + 2(.01)^2 + \cdots$.

PAGE 21 Exercise 0.5.3: We are assuming that $a \leq b$. By convention, $[a, b]$ implies $a \leq b$.

PAGE 22 First line after equation 0.6.2, replace "formed by the elements of the diagonal digits" by "formed by the diagonal digits." In the last line of the same paragraph, replace "same nth decimal" by "same nth decimal digit".

Replace the first sentence in the last paragraph before the heading "Existence of transcendental numbers" by

Infinite sets that can be put in one-to-one correspondence with the natural numbers are called *countable* or *countably infinite*.

PAGE 23 Figure 0.6.3: It seems that we misattributed the statement, "This isn't math, it's theology" to Hermite. Leonard Smiley informs us that this quotation is usually associated with Paul Gordan commenting on the Hilbert basis theorem. In an online forum, Walter Felscher of the University of Tuebingen writes that the saying is generally attributed to Gordan, but that "it does not seem to appear in Gordan's writings, and so it has the status of hearsay."

PAGE 30   Line 16: " ... and in the complex case ... "

PAGE 38   The last margin note should be on page 39.

PAGE 42   First margin note: Lewis Robinson says he uses the mnemonic "roman catholic" for "row then column".

PAGE 57   Footnote: Both affine and linear transformations give as their output first-degree polynomials in their inputs. Linear transformations are affine, of course.

PAGE 58   End of margin note: " ... and then multiplying ... ", not "the multiplying".

PAGE 65   In the 4th line of the footnote, exercise 0.6.7, not 0.6.8.

PAGE 78   2 lines before equation 1.4.47, it perhaps would have been better to omit the dot, which here denotes ordinary mutiplication, not the dot product; thus the area is $|\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\theta$.

PAGE 84   Exercise 1.4.28: We have not yet defined the trace tr for matrices bigger than 3×3. It is the sum of the diagonal entries (see definition 4.8.13).

PAGE 92   3 lines before equation 1.5.26: replace "we can define the limit $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$" by

"it makes sense to ask whether the limit $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ exists."

PAGE 97   Definition 1.5.26, last line: The second f should be bold: $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| < \epsilon$, not $|\mathbf{f}(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon$.

PAGE 99   Line 3: The codomain of a monomial function is $\mathbb{R}$.

PAGE 100   In figure 1.5.7, some of the points are mislabeled. The vector $\mathbf{a}_3$ (which should be written $\vec{\mathbf{a}}_3$) goes from $\mathbf{s}_2$ to $\mathbf{s}_3$, and so forth.

PAGE 102   Line 15: "This tells us that $\sum_k |A|^k$ converges", not "This tells us what ... ".

PAGE 108   Top of the page: Replace the last paragraph of the proof of theorem 1.6.3 by

"Consider $\mathbf{a} \in \mathbb{R}^n$ such that the mth decimal digit of each coordinate a_k agrees with the mth decimal digit of the kth coordinate of $\mathbf{x}_{i(m)}$. Then $\left|\mathbf{x}_{i(m)} - \mathbf{a}\right| < n10^{-m}$ for all m. Thus $\mathbf{x}_{i(m)}$ converges to $\mathbf{a}$. Since C is closed, $\mathbf{a}$ is in C."

PAGE 109   It is also widely believed that $\frac{1}{2\pi}$ is normal.

PAGE 109   Paragraph after example 1.6.4: Perhaps we should replace "intuitionists" by "constructivists".

PAGE 111   Proof of theorem 1.6.9, the statement at the end of the first paragraph, that if $|f(\mathbf{x}) - f(\mathbf{b})| < \epsilon$, then $|f(\mathbf{x})| < |f(\mathbf{b})| + \epsilon$, is justified by the triangle inequality:

$$|f(\mathbf{x})| = |f(\mathbf{x}) - f(\mathbf{b}) + f(\mathbf{b})| \ \le\ |f(\mathbf{x}) - f(\mathbf{b})| + |f(\mathbf{b})| \ <\ \epsilon + |f(\mathbf{b})|.$$

PAGE 114 In the last displayed equation in the margin, the $-a$ in the first line should be $-b$: $|a| = |a + b - b\,|$

PAGE 122 In the first margin note, the word "maneuverability" is misspelled.

PAGE 128 Since the gradient is a (column) vector, it would be better to use square brackets in equation 1.7.2:

$$\operatorname{grad} f(\mathbf{a}) = \vec{\nabla} f(\mathbf{a}) = \begin{bmatrix} D_1 f(\mathbf{a}) \\ \vdots \\ D_n f(\mathbf{a}) \end{bmatrix}.$$

PAGE 128 The three lines before equation 1.7.24 should be replaced by "Equation 1.7.20 says nothing about the direction in which $\vec{\mathbf{h}}$ approaches $\mathbf{0}$; it can approach $\mathbf{0}$ from any direction. In particular, we can set $\vec{\mathbf{h}} = t\vec{\mathbf{e}}_i$, and let the number t tend to 0, giving"

In equations 1.7.24 and 1.7.25, $f(\mathbf{a})$ should be $\mathbf{f}(\mathbf{a})$.

PAGE 138 In the third margin note, it might be clearer to write "By theorem 1.9.7, requiring that the first partials be differentiable is weaker than requiring that the second partials be continuous."

PAGE 141 Equation 1.8.6: The 0 in $0\vec{\mathbf{h}}$ is a zero matrix, and might be written [0]. The next two 0's are vectors, and should be bold.

PAGE 144 There should be no overset arrow in the displayed equation in the margin; $D_j(\mathbf{f} \circ \mathbf{g})_i(\mathbf{a})$ is not a vector.

PAGE 148 In equation 1.9.5, $(\mathbf{b} - \mathbf{a})$ should have an arrow over it both places it appears.

PAGE 171 Two lines before equation 2.2.9, a comma is needed: "Denote these nonzero entries by $\tilde{a}_1, \dots, \tilde{a}_k$," not "Denote ... by $\tilde{a}_1, \dots \tilde{a}_k$."

PAGE 173 Line 16: "then the set of equations will have no solutions", not " then the solution will have no solutions"

PAGE 189 In a future edition, we will add the comment that the zero vector space whose only element is the vector $\vec{\mathbf{0}}$ has only one basis, the empty set.

PAGE 213 First margin note: "preserving whatever structure is relevant", not "preserving whatever structure is relative"

PAGE 216 Next to last line, "how useful it can be to change bases", not "how useful it can to change bases ".

PAGE 219 In the caption for figure 2.6.1, $\frac{2}{3} = \frac{5}{-1}$ should be $\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

PAGE 223 Sentence before equation 2.7.3: "to compute the powers", not "to computer the powers"

PAGE 241 In the second margin note, absolute value of x_i: $|x_1| \leq \sqrt{x_1^2 + x_2^2}$.

PAGE 245 Last margin note: "a unique solution", not "one unique solution"

PAGE 266 Three lines after equation 2.10.17, $\mathbf{x}_1 = \mathbf{x}_0 + \vec{\mathbf{h}}_0(\mathbf{y})$ should be

$$\mathbf{x}_1(\mathbf{y}) = \mathbf{x}_0 + \vec{\mathbf{h}}_0(\mathbf{y}).$$

PAGE 282 Last line of exercise 2.31: "$(g(A))^2 = A$ for all", not "$(g(A))^2 = A$for all".

Exercise 2.33: 'What should you do? Suggest ways to deal with the situation", not "What should you do? ways to deal with the situation".

PAGE 303 Exercise 3.16 "in example 3.1.13", not "In example 3.1.13"

PAGE 308 Last line of text: "the equations we have been given", not "the equations we have given".

PAGE 318 Third margin note: It would be clearer to write "first partials" rather than "partials": "Our hypothesis that the first partials be differentiable".

PAGE 327 Line immediately after equation 3.4.3: $\pi \approx 3.1416$, not $\pi \approx 3.1415$.

PAGE 334 We have changed definition 3.5.1 to say that a quadratic form is a polynomial *function* :

A quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function in the variables $x_1, \dots, x_n$, all of whose terms are of degree 2.

PAGE 338 We have changed the sentence after equation 3.5.20 to read

Only the second decomposition reflects theorem 3.5.3. In the first, the three functions $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto y$, and $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x + y$ are linearly dependent.

PAGE 352 In the line immediately before equation 3.7.8, F should be $\mathbf{F}$.

PAGE 355 Two lines before equation 3.7.12, "line on that line" should be "lie on that line".

PAGE 364 First margin note, line 7: "Lagrange", not "Langrange".

PAGE 367 Margin note for exercise 3.7.11, part e: exercise 2.4.7 doesn't just say that if Q is orthogonal, then $Q^{-1} = Q^\top$; it also asks you to show the converse.

PAGE 399 Second line. "With support in $[a, b]$" means $f(x) = 0$ if $x \notin [a, b]$. If $f(x) \neq 0$ for x in some other interval, say $[c, d]$, we would say "with support in $[a, b] \cup [c, d]$".

PAGE 409 Definition 4.2.4 defines probability measure using the undefined notion of probability. Here is a more precise definition:

Definition 4.2.4 (Probability measure). The *probability measure* $\mathbf{P}$ takes an event $A \subset S$ and returns a number $\mathbf{P}(A) \in [0,1]$ such that (1) $\mathbf{P}(S) = 1$, and (2) if $\mathbf{P}(A \cap B) = \phi$ for $A, B \subset S$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$.

We say that $\mathbf{P}(A)$ corresponds to the *probability* of an outcome of the experiment being in A; it can range from 0 (it is certain that the outcome will not be in A) to 1 (it is certain that it will be in A).

Page 415 In the first margin note, the first line of the equation should have an extra set of parentheses: $E((f - E(f))^2)$, not $E(f - E(f))^2$.

Page 425 The margin notes refers to section 0.2 for uniform continuity. It is also discussed in section 1.5, in particular definition 1.5.31.

Page 472 "Clearly if A is an $n \times n$ matrix", not " … an $n \times n$ polynomial"

Page 473 First paragraph in the margin: "involve" should be "involves", and "an $n \times n$" should be "an $n \times n$ matrix".

Page 473 For proposition 4.8.22 to be true, we must allow the matrix P to be *unitary*, the complex generalization of orthogonal. This requires complex inner product, which we have not defined. They aren't really that much more difficult than real inner products, but there is a complex conjugate which shows up and makes all the computations more complicated; in our experience students find them a lot more difficult to deal with. So for pedagogical reasons, we omitted them. We have replaced the statement as follows:

Proposition 4.8.22. *If A is an $n \times n$ complex matrix, then there exists an invertible matrix P such that $P^{-1}AP$ is upper triangular. Equivalently, there is a basis $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$ such that the matrix of A in that basis is upper triangular.*

Page 496 Margin note: In the line immediately before the margin paragraph starting "It is surprising," *measure thory* should be *measure theory*.

Page 514 Margin note for exercise 4.11.18: "Part a" should be "Part b"

Page 553 5 lines before the heading "Elementary forms": "the parallelogram spanned by those k vectors" should be "the k-parallelogram spanned … ".

Page 571 Definition 6.3.1, part 1, lines 3–4: "Two forms define the same orientation" (not "same direction").

Page 594 In the remark, what we call "units", physicists would call "dimensions'. Physicists use "dimension" to describe what is being measured, as opposed to units. A force field might be measured in joules/meter or ergs/centimeter; both correspond to "dimensions" energy/length. Other cases where we use "units" rather than "dimensions" are found on pages 595, 596, 636, 640.

Page 601 Last line before the exercises: "We explore this further in section 6.9", not in section 6.8

Page 608 Two lines before equation 6.6.6: *an* $(n-k)\times n$ matrix, not "a $(n-k)\times n$ matrix"

Page 633 2nd paragraph of margin note: "better suited to insects than to people", not "... that to people""

Page 634 In a subsequent edition we will add the note

The vector fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are functions of x, y, z, and t, but the operators div and curl take derivatives only with respect to the spatial variables. Maxwell's equations as written in equations 6.9.2-6.9.5 treat space and time in different ways, and make it difficult to work in spacetime, for instance, to pass from one frame of reference to another with respect to which it is moving. By contrast, the formulation 6.9.14 of Maxwell's laws in terms of differential forms treats space and time on an equal footing.

Page 637 First line after equation 6.9.21: the constant of proportionality μ_0 is known as the *permeability* of free space.

Page 639 Three lines after equation 6.9.33: "Ampère's k_2", not "Maxwell's k_2".

Page 642 First margin note: Alternatively, one can measure the speed of light; in SI (the international system of units), the permeability μ_0 is defined to be $4\pi/10^7$, and then equation 6.9.61 says that ϵ_0 is $1/(\mu_0 c^2)$.

Page 643 Three lines after equation 6.9.56: "in free space, where there are no charges or currents" should be simply "in the absence of charges or currents".

Page 646 Similarly, in exercises 6.9.8 and 6.9.9, "in free space (no charges or currents)" should be "in the absence of charges or currents".

Page 650 Equation 6.10.22: An end parenthesis is missing after $f(x_{i+1}$.

Page 662 Exercise 6.11.16 should really be in section 6.9 or with the review exercises for chapter 6.

Page 665 Margin note: "Over an oriented curve", not over "an oriented surface".

Page 672 Exercise 6.31: In the line before the third displayed equation, "3-dimensional", not "3-dimensonal".

Page 727 Next to last margin note: "denominator in", not 'denominatorin".

Page 734 Corollary A18.1 does not follow from the new version of proposition 4.8.22.

Pages 738–740 Because we no longer can use corollary A18.1, some changes are required in the subsection "completing the proof of the change of variables formula". At the bottom of page 738, "we will denote by Z the union of these cubes" should become "We will denote by Z the union of the closure of these cubes".

The top of page 739 (up to Lemma A19.6) should become

We will also require that N_1 be large enough so that $Z \subset U$. Then $X \cup Z$ is a compact subset of U. Call M the Lipschitz ratio of $[\mathbf{D}\Phi]$, and set

$$K = \sup_{\mathbf{x} \in X \cup Z} |\det[\mathbf{D}\Phi(\mathbf{x})]| \quad \text{and} \quad L = \sup_{\mathbf{y} \in Y} |f(\mathbf{y})|. \qquad A19.25$$

We know that K is well defined because $|\det[\mathbf{D}\Phi(\mathbf{x})]|$ is continuous on $X \cup Z$.

In the remainder of the proof, K^n should be replaced by K.

Page 766 About one-third of the way down the page, the sentence about the remainder should read

the remainder satisfies $|R(f)(\vec{\mathbf{x}})| \leq C|\vec{\mathbf{x}}|^2$, for some constant C.

Page 766 6th line after equation A23.1: "the remainder is $|R(f)(\vec{\mathbf{x}})|$," not "the remainder is $|R(\vec{\mathbf{x}})|$."

Page 770 In proposition A24.2, the line before equation A24.2 should be

" ... and by $y_1, \dots, y_m$ the coordinates in $\mathbb{R}^m$. Then"

Page 771 In example A24.5 we work from the definition. As Leonard Smiley suggests, it would have been good to also give a more computational approach. Using the rules

$$\mathbf{f}^*(\varphi_1 \wedge \varphi_2) = \mathbf{f}^*\varphi_1 \wedge \mathbf{f}^*\varphi_2 \quad \text{and} \quad \mathbf{f}^*(d\varphi) = d\mathbf{f}^*\varphi,$$

we can compute:

$$\begin{aligned} \mathbf{f}^*(y_2 dy_1 \wedge dy_3) &= (x_1x_2)\big(d(x_1^2) \wedge d(x_2^2)\big) \\ &= (x_1x_2)(2x_1dx_1) \wedge (2x_2dx_2) \\ &= 4x_1^2x_2^2\, dx_1 \wedge dx_2. \end{aligned}$$

Index: corrections and additions

Page 795: ellipse, pages 481–482; electromagnetic field: page 369 not 368

Page 801: symmetric matrix, add page 55 (exercise 1.2.16) and page 304 (exercise 3.1.11), and page 313 (exercise 3.2.11). Delete page 305 (exercise 3.2.11 was moved from section 3.1 at the last minute).

Due to a last-minute change, the mention at the top of page 643 of Maxwell, Euler, Bernoulli, Lagrange, Laplace, d'Alembert, Fourier, and Poisson is incorrectly indexed as being page 642.

Page 798: in the subentry "orthogonal" (under "matrix"), page 305 should be 313. Add page 194 (the reference is to exercise 2.4.7, an important exercise).

Page 799: for the entry "orthogonal matrix", page 305 should be 313. Add page 194.

Page 802: for the entry "transpose", add page 200