For example, comparison theorems for weaker splittings of the first type are considered in Miller-Neumann [40] with the hypothesis

$$(\mathbf{A}^{-1}\mathbf{N}_1)^{i+j} \leq (\mathbf{A}^{-1}\mathbf{N}_1)^{i}(\mathbf{A}^{-1}\mathbf{N}_2)^{j}, \tag{2.128}$$

where $i \geq 1$ and $j \geq 1$.

2.5 Summary

The splittings of definition 2.1.12 extend successively a class of splittings $\mathbf{A} = \mathbf{M} - \mathbf{N}$ for which the matrices $\mathbf{N}$ and $\mathbf{M}^{-1}$ may lose the properties of nonnegativity. Distinguishing the two types of weak nonnegative splittings and the two types of weaker splittings leads to further extensions, allowing us to analyze cases where $\mathbf{M}^{-1}\mathbf{N}$ may have negative entries even if $\mathbf{N}\mathbf{M}^{-1}$ is nonnegative.

Conditions ensuring convergence of a splitting $\mathbf{A} = \mathbf{M} - \mathbf{N}$ were discussed in section 2.1 for the general case. The main result of that section, theorem 2.1.2, shows that if $\mathbf{A}$ and $\mathbf{M}$ are nonsingular matrices, then for an arbitrary splitting $\mathbf{A} = \mathbf{M} - \mathbf{N}$, the matrices $\mathbf{M}^{-1}\mathbf{N}$ and $\mathbf{A}^{-1}\mathbf{N}$ commute (as do $\mathbf{N}\mathbf{M}^{-1}$ and $\mathbf{N}\mathbf{A}^{-1}$). These commutative properties of $\mathbf{M}^{-1}\mathbf{N}$ and $\mathbf{A}^{-1}\mathbf{N}$ allowed us to determine the dependence of the eigenvalue spectra of both matrices, as shown in lemma 2.1.4.

As follows from theorem 2.3.1, the first three splittings in definition 2.1.12 are convergent if and only if $\mathbf{A}^{-1} \geq \mathbf{O}$; i.e., for these splittings,

$$\mathbf{A}^{-1} \geq \mathbf{O} \iff \varrho(\mathbf{M}^{-1}\mathbf{N}) = \varrho(\mathbf{N}\mathbf{M}^{-1}) < 1.$$

For weak and weaker splittings, the assumption $\mathbf{A}^{-1} \geq \mathbf{O}$ is not a sufficient condition for a splitting of $\mathbf{A}$ to be convergent; it is also possible to construct a convergent weak or weaker splitting when $\mathbf{A}^{-1} \not\geq \mathbf{O}$.

Moreover, the conditions $\mathbf{A}^{-1}\mathbf{N} \geq \mathbf{O}$ or $\mathbf{N}\mathbf{A}^{-1} \geq \mathbf{O}$ may not ensure that a given splitting of $\mathbf{A}$ will be weak or weaker. It follows from theorem 2.4.1 that if a splitting is weaker, then it is convergent if and only if $\mathbf{A}^{-1}\mathbf{N} \geq \mathbf{O}$ (or $\mathbf{N}\mathbf{A}^{-1} \geq \mathbf{O}$). But a convergent splitting that satisfies $\mathbf{A}^{-1}\mathbf{N} \geq \mathbf{O}$ or $\mathbf{N}\mathbf{A}^{-1} \geq \mathbf{O}$ is not necessarily weaker. Thus, the criteria for constructing convergent weak or weaker splittings of even monotone matrices remain an open question.

Comparison theorems, proved under the progressively weakening conditions presented in the scheme of condition implications shown in figure 2.2.1, provide successive generalizations, but they are accompanied by an increased complexity in the verification of these conditions.

The condition $\mathbf{N}_2 \geq \mathbf{N}_1$ and the weaker condition $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$ appear in many applications. It should be emphasized that when verifying the second condition, it is not always necessary to compute inverses: very often, the validity of this inequality can be deduced from the structure of the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ (see section 4 in [76]). This justifies considering it as a "natural" condition.

To verify the following conditions:

- the condition $\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{O}$, i.e., (E) in figure 2.2.1
- the equivalent condition $\mathbf{A}^{-1}\mathbf{M}_2 \geq \mathbf{A}^{-1}\mathbf{M}_1 \geq \mathbf{O}$, i.e., (D) in figure 2.2.1
- similar conditions in lemmas 2.3.4 and 2.4.4

- the Csordas-Varga condition $\mathbf{A}^{-1}\mathbf{N}_2\mathbf{A}^{-1} \geq \mathbf{A}^{-1}\mathbf{N}_1\mathbf{A}^{-1} \geq \mathbf{O}$, i.e., (G) in figure 2.2.1

- the conditions (H1) and (H3) used by Beauwens [8] in theorem 2.4.17

it is necessary to know the matrix $\mathbf{A}^{-1}$ explicitly, which can be a cumbersome or impractical task, when $\mathbf{A}$ has a large order. On the other hand, when $\mathbf{A}^{-1}$ is known, the solution can be obtained directly from the equation $\mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$ and convergence analysis based on the above conditions is pointless.

However, the condition (E) $\mathbf{A}^{-1}\mathbf{N}_2 \geq \mathbf{A}^{-1}\mathbf{N}_1 \geq \mathbf{O}$ (like similar conditions in lemmas 2.3.4 and 2.4.4) has an important theoretical meaning. Just showing the validity of this condition allows us to prove many comparison theorems by the relation (2.16).

Marek-Szyld [37] introduced conditions that can be verified using knowledge of eigenvectors of iteration matrices; similar conditions are used as hypotheses of comparison theorems in [76]. However, the arithmetical effort required to verify such hypotheses may be comparable to that required for an iterative or direct solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{c}$.

Csordas-Varga [14], introducing the condition

$$(\mathbf{A}^{-1}\mathbf{N}_2)^j\mathbf{A}^{-1} \geq (\mathbf{A}^{-1}\mathbf{N}_1)^j\mathbf{A}^{-1} \geq \mathbf{O}, \quad j \geq 1$$

for regular splittings, in some sense opened a new category of more complicated conditions; their work was continued by Miller-Neumann [40], who considered the hypothesis (2.128). However, they did not give any simple examples of splittings showing that the condition (2.128) is not satisfied with $i = 1$ and $j = 1$ but holds for $i > 1$ and/or $j > 1$.

Finally, in actual practice, comparison theorems based on the conditions $\mathbf{N}_2 \geq \mathbf{N}_1$ and $\mathbf{M}_1^{-1} \geq \mathbf{M}_2^{-1}$ are most often used [61, 67, 68, 76, 86].