Chapter 1

Numerical linear algebra: background

In this chapter, background material on numerical linear algebra is recalled for the convenience of the reader. Section 1.1 reviews matrix theory and defines the notion of *computational work*. Section 1.2 discusses eigenvalues and eigenvectors and what it means for a matrix to be *convergent*, *diagonally dominant*, or *irreducibly diagonally dominant*. Other topics discussed include the Perron-Frobenius theory of nonnegative matrices and the power method. Section 1.3 gives an overview of solutions to systems of linear equations.

More details can be found in the classic books by Varga [61], Young [97], Golub–Van Loan [23], and Stewart [56]. A comprehensive treatment of matrix theory and its application in electrotechnics and automatics is presented in Kaczorek's book [33].

1.1 Review of matrix theory

We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^q$ the set of real column vectors with q entries, and by $\mathbb{R}^{p\times q}$ the set of $p \times q$ real matrices.[1] Similarly, $\mathbb{C}$, $\mathbb{C}^q$, and $\mathbb{C}^{p\times q}$ denote sets of complex numbers, vectors, and matrices respectively. The set $\mathbb{R}^q$ is called *real q-dimensional space*; $\mathbb{C}^q$ is called *complex q-dimensional space.*

A lower-case letter like a or b will denote scalars. A rectangular matrix will be denoted by an upper-case bold letter such as $\mathbf{A}$, and its (i,j)th entry will be denoted by $a_{i,j}$ or $(A)_{i,j}$. For $\mathbf{A} \in \mathbb{C}^{p\times q}$ we have

$$\mathbf{A} = (a_{i,j}) = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,q} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p,1} & a_{p,2} & \cdots & a_{p,q} \end{bmatrix}, \quad a_{i,j} \in \mathbb{C}. \tag{1.1}$$

A *sparse matrix* is a matrix with a large number of zero entries; a matrix that is not

[1]Usually, matrix dimensions are denoted by $m \times n$. Since this book uses m and n as the indices of mesh points in difference approximation problems, to avoid confusion we use the letters p, q, and r to denote matrix dimensions.

sparse is *dense*.

If $q = 1$, a $p \times q$ matrix is a *column matrix* or *column vector*; if $p = 1$, the matrix is a *row matrix* or *row vector*. We denote a vector by a lower-case bold letter and the components of the vector by lower-case letters with a single subscript. Since column matrices are column vectors, we can identify $\mathbb{C}^{p\times 1}$ with $\mathbb{C}^p$: if $\mathbf{x} \in \mathbb{C}^p$, we have

$$\mathbf{x} = (x_i) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, \quad x_i \in \mathbb{C}. \tag{1.2}$$

The elements of $\mathbb{C}^{1\times q}$ are row vectors and for $\mathbf{y} \in \mathbb{C}^{1\times q}$ we have

$$\mathbf{y} = [\, y_1 \; y_2 \; \ldots \; y_q \,], \quad y_i \in \mathbb{C}.$$

If $p = q$, the matrix is said to be a *square matrix of order* q. The *diagonal* (or *main diagonal*) of a matrix $\mathbf{A} \in \mathbb{C}^{q\times q}$ consists of entries $a_{1,1}$, $a_{2,2}$, $\ldots, a_{q,q}$. A matrix is *diagonal* if all off-diagonal entries are zero. Diagonal entries below the main diagonal form a *subdiagonal*; those above the main diagonal form a *superdiagonal*.

A matrix with nonzero entries only on the main diagonal and the adjacent subdiagonal and superdiagonal is *tridagonal* (see for instance equation (3.22)); a matrix with nonzero entries only on the main diagonal and the two adjacent subdiagonals and two adjacent superdiagonals is *pentadiagonal* (see equation (3.45)). A *1-diagonal* matrix is a matrix with nonzero entries only on a single diagonal, which may or may not be the main diagonal. A matrix with nonzero entries only on n lines parallel to the main diagonal is *n-diagonal*; in the 5-diagonal matrix shown in equation (3.66), the diagonal lines are symmetric with respect to the main diagonal, but this is not required.

The diagonal matrix obtained from $\mathbf{A} \in \mathbb{C}^{q\times q}$ by replacing all off-diagonal entries of $\mathbf{A}$ by zero is denoted diag$\{\mathbf{A}\}$:

$$\operatorname{diag}\{\mathbf{A}\} = \begin{bmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & a_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & a_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{q,q} \end{bmatrix}. \tag{1.3}$$

We define

$$\text{off-diag}\{\mathbf{A}\} = \mathbf{A} - \operatorname{diag}\{\mathbf{A}\}. \tag{1.4}$$

If $a_{i,j} = 0$ for $i < j$ ($i \le j$), then $\mathbf{A} \in \mathbb{C}^{p\times p}$ is a *lower triangular* (*strictly lower triangular*) matrix. If $a_{i,j} = 0$ for $i > j$ ($i \ge j$), then $\mathbf{A} \in \mathbb{C}^{p\times p}$ is *upper triangular* (*strictly upper triangular*).

The *identity matrix of order* q, usually denoted $\mathbf{I}_q$, is a diagonal matrix with all diagonal entries equal to 1. The ith column of the identity matrix is the ith *unit vector* $\mathbf{e}_i$, so that $\mathbf{I}_q$ can be symbolically written:

$$\mathbf{I}_q = [\, \mathbf{e}_1 \; \mathbf{e}_2 \; \ldots \; \mathbf{e}_q \,]. \tag{1.5}$$

When context makes the order clear, the subscript will be dropped and we write $\mathbf{I}$ for the identity matrix.