Chapter 2

The theory of matrix splitting

The main goal of this chapter is to present the systematic analysis of convergence conditions derived from the implication scheme for the regular splitting case and used as assumptions in comparison theorems proved in the subsequent sections. The secondary goal is to survey, compare, and further develop properties of matrix splittings in order to present more clearly some aspects of results known in the literature.

Historically, the idea of matrix splittings has its origin in the regular splitting theory introduced in 1960 by Varga [61] and extended in 1973 by the author's thesis [67] (recalled in [76, 89]). These first results were given as comparison theorems for regular splittings of monotone matrices, and were proved under natural hypotheses using the Perron-Frobenius theory of nonnegative matrices. They have been useful tools in analyzing convergence of some iterative methods for solving systems of linear equations [66, 67, 68, 69, 70, 71, 72, 78, 83].

Further extensions for regular splittings were obtained by Csordas and Varga [14] in 1984; since then a renewed interest in comparison theorems, proved under progressively weaker hypotheses for different types of splittings, has been observed in the literature. These new results lead to successive generalizations, at the price of an increased complexity when it comes to verifying hypotheses. Therefore, some comparison theorems may be more significant in theory than in practice. The material presented in this chapter is mainly based on the author's results given in [76, 89].

2.1 General properties of matrix splittings

Consider the iterative solution of the linear equation system

$$\mathbf{A}\mathbf{x} = \mathbf{c}, \tag{2.1}$$

where $\mathbf{A} \in \mathbb{C}^{q\times q}$ is a nonsingular matrix and $\mathbf{x}, \mathbf{c}$ are vectors in $\mathbb{C}^q$.

Traditionally, a large class of iterative methods for solving (2.1) can be formed using the *splitting*

$$\mathbf{A} = \mathbf{M} - \mathbf{N}, \quad \text{with } \mathbf{M} \text{ nonsingular}; \tag{2.2}$$

the approximate solution $\mathbf{x}^{(t+1)}$ is generated as follows:

$$\mathbf{M}\mathbf{x}^{(t+1)} = \mathbf{N}\mathbf{x}^{(t)} + \mathbf{c}, \quad t \geq 0,$$

or, equivalently,

$$\mathbf{x}^{(t+1)} = \mathbf{M}^{-1}\mathbf{N}\mathbf{x}^{(t)} + \mathbf{M}^{-1}\mathbf{c}, \quad t \geq 0, \tag{2.3}$$

where the starting vector $\mathbf{x}^{(0)}$ is given.

It is well known (see, for instance, theorem 4.1.2) that this iterative method converges to the unique solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{c} \tag{2.4}$$

for each $\mathbf{x}^{(0)}$ if and only if $\varrho(\mathbf{M}^{-1}\mathbf{N}) < 1$, which means that the splitting $\mathbf{A} = \mathbf{M} - \mathbf{N}$ is convergent.

Definition 2.1.1 **(Convergent splitting)** Let $\mathbf{A}, \mathbf{M}, \mathbf{N}$ be elements of $\mathbb{C}^{q\times q}$, with $\mathbf{M}$ nonsingular. The decomposition $\mathbf{A} = \mathbf{M} - \mathbf{N}$ is a *convergent splitting* of $\mathbf{A}$ if

$$\varrho(\mathbf{M}^{-1}\mathbf{N}) < 1, \quad \text{or, equivalently (by lemma 1.2.6), if} \quad \varrho(\mathbf{N}\mathbf{M}^{-1}) < 1.$$

The convergence analysis of this method is based on $\varrho(\mathbf{M}^{-1}\mathbf{N})$, the spectral radius of the iteration matrix. For large values of t, the solution error decreases in magnitude approximately by a factor of $\varrho(\mathbf{M}^{-1}\mathbf{N})$ at each iteration step; the smaller $\varrho(\mathbf{M}^{-1}\mathbf{N})$, the quicker the convergence. Thus, evaluating an iterative method focuses on two issues: $\mathbf{M}$ should be an easily invertible matrix, and $\varrho(\mathbf{M}^{-1}\mathbf{N})$ should be as small as possible.

The following theorem gives general properties of a splitting $\mathbf{A} = \mathbf{M} - \mathbf{N}$ (not necessary convergent). These properties will be useful for proving comparison theorems.

Theorem 2.1.2. *Let $\mathbf{A} = \mathbf{M} - \mathbf{N}$ be a splitting of $\mathbf{A} \in \mathbb{C}^{q\times q}$. If $\mathbf{A}$ and $\mathbf{M}$ are nonsingular matrices, then*

$$\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1}, \tag{2.5}$$

the matrices $\mathbf{M}^{-1}\mathbf{N}$ and $\mathbf{A}^{-1}\mathbf{N}$ commute, and the matrices $\mathbf{N}\mathbf{M}^{-1}$ and $\mathbf{N}\mathbf{A}^{-1}$ also commute.

Proof. From the definition of a splitting of $\mathbf{A}$, it follows that

$$\mathbf{M}^{-1} = \left[\mathbf{A} + \mathbf{N}\right]^{-1} = \mathbf{A}^{-1}\left[\mathbf{I} + \mathbf{N}\mathbf{A}^{-1}\right]^{-1} = \left[\mathbf{I} + \mathbf{A}^{-1}\mathbf{N}\right]^{-1}\mathbf{A}^{-1}, \tag{2.6}$$

and

$$\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{M}^{-1} + \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1}, \tag{2.7}$$

which implies the equality (2.5) and hence,

$$\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1}\mathbf{N} = \mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1}\mathbf{N} \quad \text{and} \quad \mathbf{N}\mathbf{M}^{-1}\mathbf{N}\mathbf{A}^{-1} = \mathbf{N}\mathbf{A}^{-1}\mathbf{N}\mathbf{M}^{-1}. \qquad \square$$

Corollary 2.1.3. *Let $\mathbf{A} = \mathbf{M} - \mathbf{N}$ be a splitting of $\mathbf{A} \in \mathbb{C}^{q\times q}$. If $\mathbf{A}$ and $\mathbf{M}$ are nonsingular matrices, then by the commutative property, the matrices $\mathbf{M}^{-1}\mathbf{N}$ and $\mathbf{A}^{-1}\mathbf{N}$ have the same eigenvectors, as do $\mathbf{N}\mathbf{M}^{-1}$ and $\mathbf{N}\mathbf{A}^{-1}$.*