Chapter 4

Standard iterative methods

4.1 General theory of iterative methods

An *iterative method* generates a sequence of vectors $\{\mathbf{x}^{(t)}\},\ t = 0, 1, \ldots,$ that converges to the solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$ for a given $\mathbf{x}^{(0)}$ if and only if

$$\lim_{t\to\infty} \left\|\mathbf{x} - \mathbf{x}^{(t)}\right\| = 0.$$

In evaluating the reliability of iterative solutions of $\mathbf{A}\mathbf{x} = \mathbf{c}$, it is usually convenient to consider the *residual vector*

$$\mathbf{r}^{(t)} = \mathbf{c} - \mathbf{A}\mathbf{x}^{(t)}, \tag{4.1}$$

the *inner* (or *pseudo-residual*) *error vector*

$$\boldsymbol{\delta}^{(t)} = \mathbf{x}^{(t+1)} - \mathbf{x}^{(t)} \tag{4.2}$$

and the *true error vector*

$$\mathbf{e}^{(t)} = \mathbf{x} - \mathbf{x}^{(t)}, \tag{4.3}$$

where $\mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$ is assumed to be the "exact" solution.

Most recent iterative methods terminate as soon as the residual vector $\mathbf{r}^{(t)}$ is sufficiently small. One frequently used stopping criterion is

$$\frac{\|\mathbf{r}^{(t)}\|}{\|\mathbf{r}^{(0)}\|} < \varepsilon, \tag{4.4}$$

where ε is a prescribed number, which can be related to the true error vector $\mathbf{e}^{(t)}$ in terms of the condition number $\boldsymbol{\kappa}$ defined in (1.104): $\boldsymbol{\kappa}(\mathbf{A}) = \|\mathbf{A}\|\ \|\mathbf{A}^{-1}\|$.

LEMMA 4.1.1 Let $\mathbf{c}$, $\mathbf{x}$, and $\mathbf{x}^{(0)}$ be elements of $\mathbb{R}^q$ and let $\mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$. Then

$$\frac{\|\mathbf{e}^{(t)}\|}{\|\mathbf{e}^{(0)}\|} \le \boldsymbol{\kappa}(\mathbf{A})\frac{\|\mathbf{r}^{(t)}\|}{\|\mathbf{r}^{(0)}\|}. \tag{4.5}$$

Proof. Since

$$\mathbf{r}^{(t)} = \mathbf{c} - \mathbf{A}\mathbf{x}^{(t)} = \mathbf{A}\mathbf{e}^{(t)},$$

we have

$$\|\mathbf{e}^{(t)}\| \;=\; \|\mathbf{A}^{-1}\mathbf{A}\mathbf{e}^{(t)}\| \;\le\; \|\mathbf{A}^{-1}\|\,\|\mathbf{A}\mathbf{e}^{(t)}\| \;=\; \|\mathbf{A}^{-1}\|\,\|\mathbf{r}^{(t)}\|$$

and

$$\|\mathbf{r}^{(0)}\| \;=\; \|\mathbf{A}\mathbf{e}^{(0)}\| \;\le\; \|\mathbf{A}\|\,\|\mathbf{e}^{(0)}\|.$$

Hence,

$$\frac{\|\mathbf{e}^{(t)}\|}{\|\mathbf{e}^{(0)}\|} \le \frac{\|\mathbf{A}^{-1}\|\,\|\mathbf{r}^{(t)}\|}{\|\mathbf{A}\|^{-1}\,\|\mathbf{r}^{(0)}\|} = \boldsymbol{\kappa}(\mathbf{A})\frac{\|\mathbf{r}^{(t)}\|}{\|\mathbf{r}^{(0)}\|},$$

which completes the proof. □

The most straightforward way to solve a linear system iteratively is to reformulate $\mathbf{A}\mathbf{x} = \mathbf{c}$ as a linear fixed-point iteration

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})\mathbf{x} + \mathbf{c}, \tag{4.6}$$

from which we can define the simplest iterative method, the *Richardson iteration*:

$$\mathbf{x}^{(t+1)} = (\mathbf{I} - \mathbf{A})\mathbf{x}^{(t)} + \mathbf{c}, \quad t \ge 0. \tag{4.7}$$

The Richardson iteration is convergent if $\|\mathbf{I} - \mathbf{A}\| < 1$. In order to improve the convergence of iterative methods, a linear system is *preconditioned* by multiplying both sides of $\mathbf{A}\mathbf{x} = \mathbf{c}$ by a nonsingular matrix $\mathbf{B}$ that approximates $\mathbf{A}^{-1}$:

$$\mathbf{B}\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{c}. \tag{4.8}$$

The Richardson iteration, preconditioned with an approximate inverse $\mathbf{B}$, has the form

$$\mathbf{x}^{(t+1)} = (\mathbf{I} - \mathbf{B}\mathbf{A})\mathbf{x}^{(t)} + \mathbf{B}\mathbf{c}, \quad t \ge 0. \tag{4.9}$$

If the norm of $\mathbf{I} - \mathbf{B}\mathbf{A}$ is small, then the iteration converges rapidly, and as follows from lemma 4.1.1, terminating the iteration process based on the preconditioned residual vector $\mathbf{r}^{(t)} = \mathbf{B}\mathbf{c} - \mathbf{B}\mathbf{A}\mathbf{x}^{(t)}$ will better reflect the actual error.

The Richardson iteration is a special case of the general iterative scheme given by

$$\mathbf{x}^{(t+1)} = \mathcal{G}\mathbf{x}^{(t)} + \mathbf{g}, \quad t \ge 0, \tag{4.10}$$

where $\mathcal{G}$ is called the *iteration matrix*. Iterative methods of this form are called *linear stationary iterative methods of the first degree* because $\mathbf{x}^{(t+1)}$ does not depend on the history of the iteration process; it depends only on $\mathbf{x}^{(t)}$. Krylov methods, not discussed in this book, are nonstationary iterative methods.

The following fundamental result relates convergence to the spectral radius $\varrho(\mathcal{G})$ of the iteration matrix (see definition 1.2.3).

Theorem 4.1.2. *The iteration method (4.10) converges to the unique solution with any arbitrary choice of* $\mathbf{x}^{(0)}$ *if and only if* $\varrho(\mathcal{G}) < 1$.