Hexagonal geometry

In rectangular and triangular geometries, red-black orderings were obtained in a simple way from natural ordering by labeling mesh points alternately, as can be seen in figures 3.4.15 and 3.4.17. The reduced formulas (3.116) and (3.121) were derived from the original formulas by eliminating the unknowns at coupled neighboring points; in each reduced formula, the mesh point coupling does not involve points coupled by the original formula.

This approach cannot be used in hexagonal geometry, because of the mesh structure of hexagonal geometry. However, combining a 7-point formula with 4-point formulas provides a 7-point reduced formula that turns out to be a very efficient computational tool in applications.

To simplify the discussion, but without loss of generality, we can assume that all triangles in the hexagonal cell shown in figure 3.4.10 have the same physical properties, so that $D_i = D_0$ and $\Sigma_i = \Sigma_0$ for all $i = 1, 2, \ldots, 6$. Then the 7-point formula (3.103) can be reformulated as

$$k_0 \phi_0 = c_0 + \sum_{i=1}^{6} \phi_i, \tag{3.122}$$

where

$$k_0 = 6 + \frac{3h^2 \Sigma_0}{2D_0} \quad \text{and} \quad c_0 = \frac{3h^2 S_0}{2D_0}.$$

This difference approximation is called the *H-II difference scheme*; its mesh point coupling is shown in figure 3.4.19 (left); the area of integration is the hexagon with vertical dotted lines. Evidently, when formula (3.122) is used for the mesh point 1, coupled with points connected by dotted lines in figure 3.4.19 (left), this formula must be modified by using the average values of the coefficients D and Σ at the point 1, because it is a corner point for three hexagons with different physical properties. In the general case, where six triangles with different physical properties meet at a given mesh point, as shown in figure 3.4.10, instead of D_0 we can use $\overline{D}_0 = \frac{1}{6}\sum_{i=1}^{6} D_i$ and $\overline{\Sigma}_0 = \frac{1}{6}\sum_{i=1}^{6} \Sigma_i$.

The mesh-centered 7-point formula (3.107) can be similarly reformulated as follows:

$$\bar{k}_0 \phi_0 = \bar{c}_0 + \sum_{i=I}^{VI} \frac{2D_i}{D_0 + D_i} \phi_i, \tag{3.123}$$

where

$$\bar{k}_0 = \sum_{i=I}^{VI} \frac{2D_i}{D_0 + D_i} + \frac{9h^2 \Sigma_0}{2D_0} \quad \text{and} \quad \bar{c}_0 = \frac{9h^2 S_0}{2D_0}.$$

The mesh point coupling in this formula, called the *H-II-O difference scheme*, is shown in figure 3.4.19 (right); the area of integration is the hexagon marked by vertical dotted lines. The area of integration for such a formula at the point "I" is marked by horizontal dotted lines.

Marakazov [70] proposed a reduced system formula based on eliminating in (3.122)

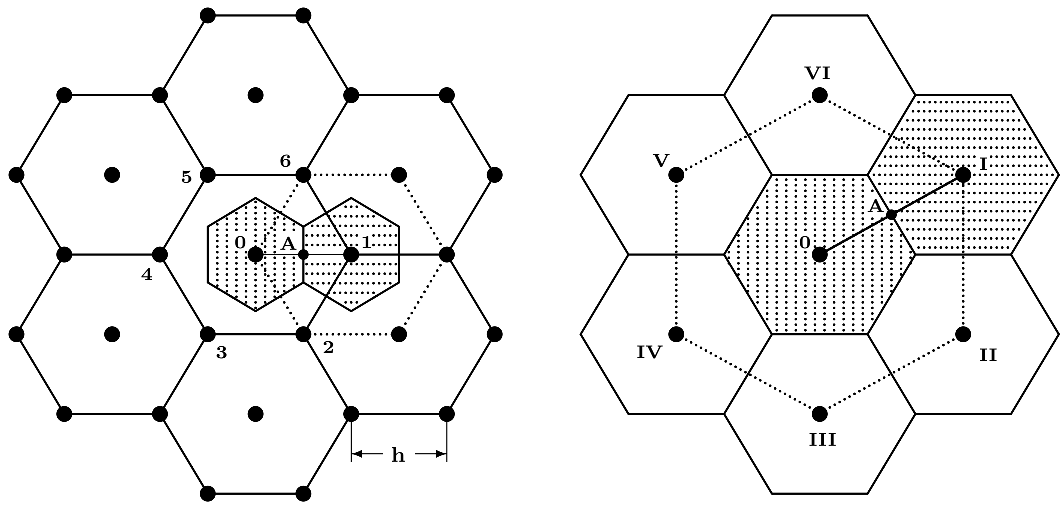

FIGURE 3.4.19. LEFT: H-II difference scheme. RIGHT: H-II-0 difference scheme. In both cases the area of integration is the hexagon with vertical dotted lines.

the unknowns at points 1, 2, ..., 6, using the following mesh-edged 4-point formulas:

$$\left.\begin{aligned} k_1\phi_1 &= c_1 + D_0\phi_0 + D_I\phi_I + D_{II}\phi_{II}, \\ k_2\phi_2 &= c_2 + D_0\phi_0 + D_{II}\phi_{II} + D_{III}\phi_{III}, \\ &\vdots \\ k_5\phi_5 &= c_5 + D_0\phi_0 + D_V\phi_V + D_{VI}\phi_{VI}, \\ k_6\phi_6 &= c_6 + D_0\phi_0 + D_{VI}\phi_{VI} + D_I\phi_I, \end{aligned}\right\} \qquad (3.124)$$

where

$$k_1 = \frac{h^2}{4}(\Sigma_0 + \Sigma_I + \Sigma_{II}) + D_0 + D_I + D_{II} \quad \text{and } c_1 = \frac{h^2}{4}(S_0 + S_I + S_{II}),$$
$$k_2 = \frac{h^2}{4}(\Sigma_0 + \Sigma_{II} + \Sigma_{III}) + D_0 + D_{II} + D_{III} \quad \text{and } c_2 = \frac{h^2}{4}(S_0 + S_{II} + S_{III}),$$
$$\vdots \qquad\qquad \vdots$$
$$k_5 = \frac{h^2}{4}(\Sigma_0 + \Sigma_V + \Sigma_{VI}) + D_0 + D_V + D_{VI} \quad \text{and } c_5 = \frac{h^2}{4}(S_0 + S_V + S_{VI}),$$
$$k_6 = \frac{h^2}{4}(\Sigma_0 + \Sigma_{VI} + \Sigma_I) + D_0 + D_{VI} + D_I \quad \text{and } c_6 = \frac{h^2}{4}(S_0 + S_{VI} + S_I).$$

The mesh point coupling of such a mesh-edged 4-point formula for the point labeled 1 is shown in figure 3.4.20 (left); the area of integration is the triangular region marked by horizontal dotted lines.

Substituting the unknowns $\phi_1, \phi_2, \ldots, \phi_6$, derived from equations (3.124)–(3.122),