Preface

Joseph Fourier (1768–1830)

Fourier was arrested during the French Revolution and threatened with the guillotine, but survived and later accompanied Napoleon to Egypt; in his day he was as well known for his studies of Egypt as for his contributions to mathematics and physics. He found a way to solve linear partial differential equations while studying heat diffusion. An emphasis on computationally effective algorithms is one theme of this book.

. . . The numerical interpretation . . . is however necessary. . . . So long as it is not obtained, the solutions may be said to remain incomplete and useless, and the truth which it is proposed to discover is no less hidden in the formulae of analysis than it was in the physical problem itself.

–Joseph Fourier, *The Analytic Theory of Heat*

Chapters 1 through 6 of this book cover the standard topics in multivariate calculus and a first course in linear algebra. The book can also be used for a course in analysis, using the proofs in the Appendix.

The organization and selection of material differs from the standard approach in three ways, reflecting the following principles.

First, we believe that at this level linear algebra should be more a convenient setting and language for multivariate calculus than a subject in its own right. The guiding principle of this unified approach is that locally, a nonlinear function behaves like its derivative.

When we have a question about a nonlinear function we answer it by looking carefully at a linear transformation: its derivative. In this approach, everything learned about linear algebra pays off twice: first for understanding linear equations, then as a tool for understanding nonlinear equations. We discuss abstract vector spaces in Section 2.6, but the emphasis is on $\mathbb{R}^n$, as we believe that most students find it easiest to move from the concrete to the abstract.

Second, we emphasize computationally effective algorithms, and we prove theorems by showing that these algorithms work.

We feel this better reflects the way mathematics is used today, in applied and pure mathematics. Moreover, it can be done with no loss of rigor. For linear equations, row reduction is the central tool; we use it to prove all the standard results about dimension and rank. For nonlinear equations, the cornerstone is Newton's method, the best and most widely used method for solving nonlinear equations; we use it both as a computational tool and in proving the inverse and implicit function theorems. We include a section on numerical methods of integration, and we encourage the use of computers both to reduce tedious calculations and as an aid in visualizing curves and surfaces.

Third, we use differential forms to generalize the fundamental theorem of calculus to higher dimensions.

The great conceptual simplification gained by doing electromagnetism in the language of forms is a central motivation for using forms. We apply the language of forms to electromagnetism and potentials in Sections 6.12 and 6.13.

In our experience, differential forms can be taught to freshmen and sophomores if forms are presented geometrically, as integrands that take an oriented piece of a curve, surface, or manifold, and return a number. We are aware that students taking courses in other fields need to master the language of vector calculus, and we devote three sections of Chapter 6 to integrating the standard vector calculus into the language of forms.

Other significant ways this book differs from standard texts include

- ◇ Applications involving big matrices
- ◇ The treatment of eigenvectors and eigenvalues
- ◇ Lebesgue integration
- ◇ Rules for computing Taylor polynomials

A few minutes spent on the Internet finds a huge range of applications of principal component analysis.

Big data Example 2.7.12 discussing the Google PageRank algorithm shows the power of the Perron-Frobenius theorem. Example 3.8.10 illustrates an application of principal component analysis, which is built on the singular value decomposition.

Eigenvectors and eigenvalues In keeping with our prejudice in favor of computationally effective algorithms, we provide in Section 2.7 a theory of eigenvectors and eigenvalues that bypasses determinants, which are more or less uncomputable for large matrices. This treatment is also stronger theoretically: Theorem 2.7.9 gives an "if and only if" statement for the existence of eigenbases. In addition, our emphasis on defining an eigenvector $\mathbf{v}$ as satisfying $A\mathbf{v} = \lambda\mathbf{v}$ has the advantage of working when A is a linear transformation between infinite-dimensional vector spaces, whereas the definition in terms of roots of the characteristic polynomial does not. However, in Section 4.8 we define the characteristic polynomial of a matrix, connecting eigenvalues and eigenvectors to determinants.

In our experience, undergraduates, even freshmen, are quite prepared to approach the Lebesgue integral via the Riemann integral, but the approach via measurable sets and σ-algebras of measurable sets is inconceivable.

Lebesgue integration We give a new approach to Lebesgue integration, tying it much more closely to Riemann integrals. We had two motivations. First, integrals over unbounded domains and integrals of unbounded functions are really important, for instance in physics and probability, and students will need to know about such integrals before they take a course in analysis. Second, there simply does not appear to be a successful theory of improper multiple integrals.

Rules for computing Taylor polynomials Even good graduate students are often unaware of the rules that make computing Taylor polynomials in higher dimensions palatable. We give these in Section 3.4.

How the book has evolved: the first four editions

The first edition of this book, published by Prentice Hall in 1999, was a mere 687 pages. The basic framework of our guiding principles was there, but we had no Lebesgue integration and no treatment of electromagnetism.

The second edition, published by Prentice Hall in 2002, grew to 800 pages. The biggest change was replacing improper integrals by Lebesgue integrals. We also added approximately 270 new exercises and 50 new examples and reworked the treatment of orientation. This edition first saw the inclusion of photos of mathematicians.

In September 2006 we received an email from Paul Bamberg, senior lecturer on mathematics at Harvard University, saying that Prentice Hall had declared the book out of print (something Prentice Hall had neglected to mention to us). We obtained the copyright and set to work on the third edition. We put exercises in small type, freeing up space for quite a bit of new material, including a section on electromagnetism; a discussion of eigenvectors, eigenvalues, and diagonalization; a discussion of the determinant and eigenvalues; and a section on integration and curvature.

The major impetus for the fourth edition (2009) was that we finally hit on what we consider the right way to define orientation of manifolds. We also expanded the page count to 818, which made it possible to add a proof of Gauss's *theorem egregium*, a discussion of Faraday's experiments, a trick for finding Lipschitz ratios for polynomial functions, a way to classify constrained critical points using the augmented Hessian matrix, and a proof of Poincaré's lemma for arbitrary forms, using the cone operator.

What's new in the fifth edition

The initial impetus for producing a new edition rather than reprinting the fourth edition was to reconcile differences in numbering (propositions, examples, etc.) in the first and second printings of the fourth edition.

An additional impetus came from discussions John Hubbard had at an AMS meeting in Washington, DC, in October 2014. Mathematicians and computer scientists there told him that existing textbooks lack examples of "big matrices". This led to two new examples illustrating the power of linear algebra and calculus: Example 2.7.12, showing how Google uses the Perron-Frobenius theorem to rank web pages, and Example 3.8.10, showing how the singular value decomposition (Theorem 3.8.1) can be used for computer face recognition.

The more we worked on the new edition, the more we wanted to change. An inclusive list is impossible; here are some additional highlights.

- ◇ In several places in the fourth edition (for instance, the proof of Proposition 6.4.8 on orientation-preserving parametrizations) we noted that "in Chapter 3 we failed to define the differentiability of functions defined on manifolds, and now we pay the price". For this edition we "paid the price" (Proposition and Definition 3.2.9) and the effort paid off handsomely, allowing us to shorten and simplify a number of proofs.
- ◇ We rewrote the discussion of multi-index notation in Section 3.3.

- ◇ We rewrote the proof of Stokes's theorem and moved most of it out of the Appendix and into the main text.
- ◇ We added Example 2.4.17 on Fourier series.
- ◇ We rewrote the discussion of classifying constrained critical points.
- ◇ We use differentiation under the integral sign to compute the Fourier transform of the Gaussian, and discuss its relation to the Heisenberg uncertainty principle.
- ◇ We added a new proposition (2.4.18) about orthonormal bases.
- ◇ We greatly expanded the discussion of orthogonal matrices.
- ◇ We added a new section in Chapter 3 on finite probability, showing the connection between probability and geometry. The new section also includes the statement and proof of the singular value decomposition.
- ◇ We added about 40 new exercises.

A student solution manual, with solutions to odd-numbered exercises, is available from Matrix Editions. Instructors who wish to acquire the instructors' solution manual should write hubbard@matrixeditions.com.

Practical information

Chapter 0 and back cover Chapter 0 is intended as a resource. We recommend that students skim through it to see if any material is unfamiliar. The inside back cover lists some useful formulas.

Errata Errata will be posted at

http://www.MatrixEditions.com

Exercises Exercises are given at the end of each section; chapter review exercises are given at the end of each chapter, except Chapter 0 and the Appendix. Exercises range from very easy exercises intended to make students familiar with vocabulary, to quite difficult ones. The hardest exercises are marked with an asterisk (in rare cases, two asterisks).

Jean Dieudonné (1906–1992)

Dieudonné, one of the founding members of "Bourbaki", a group of young mathematicians who published collectively under the pseudonym Nicolas Bourbaki, and whose goal was to put modern mathematics on a solid footing, was the personification of rigor in mathematics. Yet in his book *Infinitesimal Calculus* he put the harder proofs in small type, saying "a beginner will do well to accept plausible results without taxing his mind with subtle proofs."

Notation Mathematical notation is not always uniform. For example, $|A|$ can mean the length of a matrix A (the usage in this book) or the determinant of A (which we denote by $\det A$). Different notations for partial derivatives also exist. This should not pose a problem for readers who begin at the beginning and end at the end, but for those who are using only selected chapters, it could be confusing. Notations used in the book are listed on the front inside cover, along with an indication of where they are first introduced.

In this edition, we have changed the notation for sequences: we now denote sequences by "$i \mapsto x_i$" rather than "x_i" or "$x_1, x_2, \dots$". We are also more careful to distinguish between equalities that are true by definition, denoted $\stackrel{\text{def}}{=}$, and those true by reasoning, denoted $=$. But, to avoid too heavy notation, we write $=$ in expressions like "set $\mathbf{x} = \gamma(\mathbf{u})$".

Numbering Theorems, lemmas, propositions, corollaries, and examples share the same numbering system: Proposition 2.3.6 is not the sixth proposition of Section 2.3; it is the sixth numbered item of that section.

Figures and tables share their own numbering system; Figure 4.5.2 is the second figure or table of Section 4.5. Virtually all displayed equations and inequalities are numbered, with the numbers given at right; equation 4.2.3 is the third equation of Section 4.2.

We often refer back to theorems, examples, and so on, and believe this numbering makes them easier to find.

Readers are welcome to propose additional programs (or translations of these programs into other programming languages); if interested, please write John Hubbard at jhh8@cornell.edu.

Programs The Newton.m program used in Section 2.8 works in Matlab; it is posted at http://MatrixEditions.com/Programs.html. (Two other programs available there, Monte Carlo and Determinant, are written in Pascal and probably no longer usable.)

Symbols We use △ to mark the end of an example or remark, and □ to mark the end of a proof. Sometimes we specify what proof is being ended: □ Corollary 1.6.16 means "end of the proof of Corollary 1.6.16".

Using this book as a calculus text or as an analysis text

This book can be used at different levels of rigor. Chapters 1 through 6 contain material appropriate for a course in linear algebra and multivariate calculus. Appendix A contains the technical, rigorous underpinnings appropriate for a course in analysis. It includes proofs of those statements not proved in the main text, and a painstaking justification of arithmetic.

In deciding what to include in this appendix, and what to put in the main text, we used the analogy that learning calculus is like learning to drive a car with standard transmission – acquiring the understanding and intuition to shift gears smoothly when negotiating hills, curves, and the stops and starts of city streets. Analysis is like designing and building a car. To use this book to "learn how to drive", Appendix A should be omitted.

The SAT test used to have a section of analogies; the "right" answer sometimes seemed contestable. In that spirit,

Calculus is to analysis as playing a sonata is to composing one.

Calculus is to analysis as performing in a ballet is to choreographing it.

Analysis involves more painstaking technical work, which at times may seem like drudgery, but it provides a level of mastery that calculus alone cannot give.

Most of the proofs included in this appendix are more difficult than the proofs in the main text, but difficulty was not the only criterion; many students find the proof of the fundamental theorem of algebra (Section 1.6) quite difficult. But we find this proof qualitatively different from the proof of the Kantorovich theorem, for example. A professional mathematician who has understood the proof of the fundamental theorem of algebra should be able to reproduce it. A professional mathematician who has read through the proof of the Kantorovich theorem, and who agrees that each step is justified, might well want to refer to notes in order to reproduce it. In this sense, the first proof is more conceptual, the second more technical.

One-year courses

At Cornell University this book is used for the honors courses Math 2230 and 2240. Students are expected to have a 5 on the Advanced Placement BC Calculus exam, or the equivalent. When John Hubbard teaches the course, he typically gets to the middle of Chapter 4 in the first semester, sometimes skipping Section 3.9 on the geometry of curves and surfaces, and going through Sections 4.2–4.4 rather rapidly, in order to get to Section 4.5 on Fubini's theorem and begin to compute integrals. In the second semester he gets to the end of Chapter 6 and goes on to teach some of the material that will appear in a sequel volume, in particular differential equations.[1]

One could also spend a year on Chapters 1–6. Some students might need to review Chapter 0; others may be able to include some proofs from the appendix.

Semester courses

1. A semester course for students who have had a solid course in linear algebra

We used an earlier version of this text with students who had taken a course in linear algebra, and feel they gained a great deal from seeing how linear algebra and multivariate calculus mesh. Such students could be expected to cover chapters 1–6, possibly omitting some material. For a less fast-paced course, the book could also be covered in a year, possibly including some proofs from the appendix.

2. A semester course in analysis for students who have studied multivariable calculus

In one semester one could hope to cover all six chapters and some or most of the proofs in the appendix. This could be done at varying levels of difficulty; students might be expected to follow the proofs, for example, or they might be expected to understand them well enough to construct similar proofs.

Use by graduate students

Many graduate students have told us that they found the book very useful in preparing for their qualifying exams.

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John H. Hubbard (BA Harvard University, PhD University of Paris) is professor of mathematics at Cornell University and professor emeritus at the Université Aix-Marseille; he is the author of several books on differential equations (with Beverly West), a book on Teichmüller theory, and a two-volume book in French on scientific computing (with Florence Hubert). His research mainly concerns

[1]Eventually, he would like to take three semesters to cover chapters 1–6 of the current book and material in the forthcoming sequel, including differential equations, inner products (with Fourier analysis and wavelets), and advanced topics in differential forms.

complex analysis, differential equations, and dynamical systems. He believes that mathematics research and teaching are activities that enrich each other and should not be separated.

Barbara Burke Hubbard (BA Harvard University) is the author of *The World According to Wavelets*, which was awarded the prix d'Alembert by the French Mathematical Society in 1996. She founded Matrix Editions in 2002.