6.9.2 In the text we proved Proposition 6.9.7 in the special case where the mapping $\mathbf{f}$ is linear. Prove the general statement, where $\mathbf{f}$ is only assumed to be of class C^1.

6.9.3 Let $U \subset \mathbb{R}^n$ be open, $f: U \to \mathbb{R}^m$ of class C^1, and ξ a vector field on $\mathbb{R}^m$.

a. Show that $(f^* W_\xi)(\mathbf{x}) = W_{[\mathbf{D}f(\mathbf{x})]^\top \xi(\mathbf{x})}$.

b. Let $m = n$. Show that if $[\mathbf{D}f(\mathbf{x})]$ is invertible, then

$$(f^*\Phi_\xi)(\mathbf{x}) = \det[\mathbf{D}f(\mathbf{x})]\Phi_{[\mathbf{D}f(\mathbf{x})]^{-1}\xi(\mathbf{x})}.$$

c. Let $m = n$, let A be a square matrix, and let $A_{[j,i]}$ be the matrix obtained from A by erasing the jth row and the ith column. Let $\operatorname{adj}(A)$ be the matrix whose (i,j)th entry is $(\operatorname{adj}(A))_{i,j} = (-1)^{i+j} \det A_{[j,i]}$. Show that

$$A(\operatorname{adj}(A)) = (\det A)I \quad \text{and} \quad f^*\Phi_\xi(\mathbf{x}) = \Phi_{\operatorname{adj}([\mathbf{D}f(\mathbf{x})])\xi(\mathbf{x})}$$

Exercise 6.9.3: The matrix $\operatorname{adj}(A)$ of part c is the *adjoint matrix* of A. The equation in part b is unsatisfactory: it does not say how to represent $(f^*\Phi_\xi)(\mathbf{x})$ as the flux of a vector field when the $n \times n$ matrix $[\mathbf{D}f(\mathbf{x})]$ is not invertible. Part c deals with this situation.

6.10 The generalized Stokes's theorem

We worked hard to define the exterior derivative and to define orientation of manifolds and of boundaries. Now we are going to reap some rewards for our labor: a higher-dimensional analogue of the fundamental theorem of calculus, Stokes's theorem. It covers in one statement the four integral theorems of vector calculus, which are explored in Section 6.11.

Recall the fundamental theorem of calculus:

Theorem 6.10.1 (Fundamental theorem of calculus). *If f is a C^1 function on a neighborhood of $[a,b]$, then*

$$\int_a^b f'(t)\,dt = f(b) - f(a). \qquad 6.10.1$$

Restate this as

$$\int_{[a,b]} \mathbf{d}f = \int_{\partial[a,b]} f, \qquad 6.10.2$$

i.e., the integral of $\mathbf{d}f$ over the oriented interval $[a,b]$ is equal to the integral of f over the oriented boundary $+b - a$ of the interval. This is the case $k = n = 1$ of Theorem 6.10.2.

Special cases of the generalized Stokes's theorem are discussed in Section 6.11.

Theorem 6.10.2 is probably the best tool mathematicians have for deducing global properties from local properties. It is a wonderful theorem. It is often called the generalized Stokes's theorem, to distinguish it from the special case (surfaces in $\mathbb{R}^3$) also known as Stokes's theorem.

Theorem 6.10.2 (Generalized Stokes's theorem). *Let X be a piece-with-boundary of a k-dimensional oriented smooth manifold M in $\mathbb{R}^n$. Give the boundary ∂X of X the boundary orientation, and let φ be a $(k-1)$-form of class C^2 defined on an open set containing X. Then*

$$\int_{\partial X} \varphi = \int_X \mathbf{d}\varphi. \qquad 6.10.3$$

This beautiful, short statement is the main result of the theory of forms. Note that the dimensions in equation 6.10.3 make sense: if X is k-dimensional, ∂X is $(k-1)$-dimensional, and if φ is a $(k-1)$-form, $\mathbf{d}\varphi$ is a k-form, so $\mathbf{d}\varphi$ can be integrated over X, and φ can be integrated over ∂X.

You apply Stokes's theorem every time you use antiderivatives to compute an integral: to compute the integral of the 1-form $f\,dx$ over the oriented line segment $[a, b]$, you begin by finding a function g such that $\mathbf{d}g = f\,dx$, and then say

$$\int_a^b f\,dx = \int_{[a,b]} \mathbf{d}g = \int_{\partial[a,b]} g = g(b) - g(a). \qquad 6.10.4$$

This isn't quite the way Stokes's theorem is usually used in higher dimensions, where "looking for antiderivatives" has a different flavor.

Example 6.10.3 (Integrating over the boundary of a square). Let S be the square described by the inequalities $|x|, |y| \le 1$, with the standard orientation. To compute the integral $\int_C x\,dy - y\,dx$, where C is the boundary of S, with the boundary orientation, one possibility is to parametrize the four sides of the square (being careful to get the orientations right), then to integrate $x\,dy - y\,dx$ over all four sides and add. Another possibility is to apply Stokes's theorem:

$$\int_C x\,dy - y\,dx = \int_S \mathbf{d}(x\,dy - y\,dx) = \int_S 2\,dx \wedge dy = \int_S 2|dx\,dy| = 8. \qquad 6.10.5$$

(The square S has sidelength 2, so its area is 4.) △

What is the integral over C of $x\,dy + y\,dx$? Check below.[12]

Example 6.10.4 (Integrating over the boundary of a cube). Let us integrate the 2-form

$$\varphi = (x - y^2 + z^3)\,(dy \wedge dz + dx \wedge dz + dx \wedge dy) \qquad 6.10.6$$

over the boundary of the cube C_a given by $0 \le x, y, z \le a$.

It is quite possible to do this directly, parametrizing all six faces of the cube, but Stokes's theorem simplifies things substantially.

Computing the exterior derivative of φ gives

$$\begin{aligned} \mathbf{d}\varphi &= dx \wedge dy \wedge dz - 2y\,dy \wedge dx \wedge dz + 3z^2\,dz \wedge dx \wedge dy \\ &= (1 + 2y + 3z^2)\,dx \wedge dy \wedge dz, \end{aligned} \qquad 6.10.7$$

[12] $\mathbf{d}(x\,dy + y\,dx) = dx \wedge dy + dy \wedge dx = 0$, so the integral is 0.

Theorem 6.10.2: A lot is hidden in the equation

$$\int_{\partial X} \varphi = \int_X \mathbf{d}\varphi.$$

First, we can only integrate forms over manifolds, and the boundary of X is not a manifold, so despite its elegance, the equation does not make sense. What we mean is

$$\int_{\partial^s_M X} \varphi = \int_X \mathbf{d}\varphi;$$

by Proposition 6.6.3 and Definition 6.6.21, the smooth boundary is an oriented manifold. Since by the definition of a piece-with-boundary the set of nonsmooth points in the boundary has $(k-1)$-dimensional volume 0, we can set

$$\int_{\partial X} \varphi \overset{\text{def}}{=} \int_{\partial^s_M X} \varphi.$$

Second, the smooth boundary will usually not be compact, so how do we know the integral is defined (that it doesn't diverge)? This is why we require the second condition of a piece-with-boundary: that its smooth boundary have finite $(k-1)$-dimensional volume.

Recall (Definition 6.6.8) that a piece-with-boundary is compact. Without that hypothesis, Stokes's theorem would be false: the proof uses Heine-Borel (Theorem A3.3), which applies to compact subsets of $\mathbb{R}^n$.

For the proof of Stokes's theorem, the manifold M must be at least of class C^2; in practice, it will always be of class C^∞.