5

Volumes of manifolds

5.0 Introduction

In Chapter 4 we saw how to integrate over subsets of $\mathbb{R}^n$, first using dyadic pavings, then more general pavings. But these subsets were flat n-dimensional subsets of $\mathbb{R}^n$. What if we want to integrate over a (curvy) surface in $\mathbb{R}^3$? Many situations of obvious interest, like the area of a surface, or the total energy stored in the surface tension of a soap bubble, or the amount of fluid flowing through a pipe, are clearly some sort of surface integral. In a physics course you may have learned that the *electric flux* through a closed surface is proportional to the electric charge inside that surface.

When we say that in Chapter 4 we had "flat domains" we mean we had n-dimensional subsets of $\mathbb{R}^n$. A disc in the plane is flat, even though its boundary is a circle: we cannot bend a disc and have it remain a subset of the plane. A subset of $\mathbb{R}$ is necessarily straight; if we want a wiggly line we must allow for at least two dimensions.

In this chapter we will show how to compute the area of a surface in $\mathbb{R}^3$, or more generally, the k-dimensional volume of a k-manifold in $\mathbb{R}^n$, where $k < n$. We can't use the approach given in Section 4.1, where we saw that when integrating over a subset $A \subset \mathbb{R}^n$,

$$\int_A g(\mathbf{x})\,|d^n\mathbf{x}| = \int_{\mathbb{R}^n} g(\mathbf{x})\mathbf{1}_A(\mathbf{x})\,|d^n\mathbf{x}|. \tag{4.1.8}$$

If we try to use this equation to integrate a function in $\mathbb{R}^3$ over a surface, the integral will certainly vanish, since the surface has three-dimensional volume 0. For any k-manifold M embedded in $\mathbb{R}^n$, with $k < n$, the integral would certainly vanish, since M has n-dimensional volume 0. Instead, we need to rethink the whole process of integration.

At heart, integration is always the same:

> *Break up the domain into little pieces, assign a little number to each little piece, and finally add together all the numbers.* Then break the domain into littler pieces and repeat, taking the limit as the decomposition becomes infinitely fine. The *integrand* is the thing that assigns the number to the little piece of the domain.

There is quite a lot of leeway when choosing what kind of "little pieces" to use; choosing a decomposition of a surface into little pieces is analogous to choosing a paving, and as we saw in Section 4.7, there are many possible choices besides the dyadic paving.

The words "little piece" in this heuristic description need to be pinned down before we can do anything useful. We will break the domain into k-dimensional parallelograms, and the "little number" we attach to each little parallelogram will be its k-dimensional volume. In Section 5.1 we will see how to compute this volume.

In Chapter 6 we will study a different kind of integrand, which assigns numbers to little pieces of *oriented* manifolds.

We can only integrate over parametrized domains, and if we use the definition of parametrizations given in Chapter 3, we will not be able to parametrize even such simple objects as the circle. Section 5.2 gives a looser definition of parametrization, sufficient for integration. In Section 5.3 we compute volumes of k-manifolds; in Section 5.4 we prove three theorems relating curvature to integration, including two results that explain the meaning of Gaussian and mean curvature. Fractals and fractional dimension are discussed in Section 5.5.

5.1 Parallelograms and their volumes

We saw in Section 4.9 that the volume of a k-parallelogram in $\mathbb{R}^k$ is

$$\text{vol}_k\, P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = |\det[\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]|. \qquad 5.1.1$$

What about a k-parallelogram in $\mathbb{R}^n$? Clearly if we draw a parallelogram on a rigid piece of cardboard, cut it out, and move it about in space, its area will not change. This area should depend only on the lengths of the vectors spanning the parallelogram and the angle between them, not on where they are placed in $\mathbb{R}^3$. It isn't obvious how to compute this volume: equation 5.1.1 can't be applied, since the determinant only exists for square matrices. A formula involving the cross product (see Proposition 1.4.20) exists for a 2-parallelogram in $\mathbb{R}^3$. How will we compute the area of a 2-parallelogram in $\mathbb{R}^4$, never mind a 3-parallelogram in $\mathbb{R}^5$?

The following proposition is the key. It concerns k-parallelograms in $\mathbb{R}^k$, but we will be able to apply it to k-parallelograms in $\mathbb{R}^n$.

Exercise 5.1.3 asks you to show that if $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ are linearly dependent, then

$$\text{vol}_k(P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)) = 0.$$

In particular, this shows that if $k > n$, then

$$\text{vol}_k(P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)) = 0.$$

Proposition 5.1.1 (Volume of a k-parallelogram in $\mathbb{R}^k$). *Let $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ be k vectors in $\mathbb{R}^k$, so that $T = [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]$ is a square $k \times k$ matrix. Then*

$$\text{vol}_k\, P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \sqrt{\det(T^\top T)}. \qquad 5.1.2$$

Proof of Proposition 5.1.1: Recall that if A and B are $n \times n$ matrices, then

$$\det A \det B = \det(AB)$$
$$\det A = \det A^\top$$

(Theorems 4.8.4 and 4.8.8).

Proof. $\sqrt{\det(T^\top T)} = \sqrt{(\det T^\top)(\det T)} = \sqrt{(\det T)^2} = |\det T|$ □

Example 5.1.2 (Volume of two-dimensional and three-dimensional parallelograms). When $k = 2$, we have

$$\det(T^\top T) = \det\left(\begin{bmatrix} \vec{\mathbf{v}}_1^\top \\ \vec{\mathbf{v}}_2^\top \end{bmatrix} \begin{bmatrix} \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 \end{bmatrix}\right) = \det \begin{bmatrix} |\vec{\mathbf{v}}_1|^2 & \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 \\ \vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_1 & |\vec{\mathbf{v}}_2|^2 \end{bmatrix}$$
$$= |\vec{\mathbf{v}}_1|^2 |\vec{\mathbf{v}}_2|^2 - (\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2)^2. \qquad 5.1.3$$

Recall (Definition 1.4.6) that

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = |\vec{\mathbf{x}}|\,|\vec{\mathbf{y}}| \cos\alpha,$$

where α is the angle between the vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.

If we write $\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = |\vec{\mathbf{v}}_1||\vec{\mathbf{v}}_2| \cos\theta$ (where θ is the angle between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$), this becomes

$$\det(T^\top T) = |\vec{\mathbf{v}}_1|^2 |\vec{\mathbf{v}}_2|^2 (1 - \cos^2\theta) = |\vec{\mathbf{v}}_1|^2 |\vec{\mathbf{v}}_2|^2 \sin^2\theta. \qquad 5.1.4$$

Thus Proposition 5.1.1 asserts that the area of the 2-parallelogram spanned by $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ is

$$\sqrt{\det(T^\top T)} = |\vec{\mathbf{v}}_1||\vec{\mathbf{v}}_2||\sin\theta|, \qquad 5.1.5$$

which agrees with the formula of height times base given in high school: if $\vec{\mathbf{v}}_2$ is the base, the height is $\vec{\mathbf{v}}_1 \sin\theta$.

The same computation in the case $k = 3$ leads to a much less familiar formula. Suppose $T = [\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$, and that the angle between $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$ is θ_1, that between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_3$ is θ_2, and that between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ is θ_3. Then

$$T^\top T = \begin{bmatrix} |\vec{\mathbf{v}}_1|^2 & \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 & \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_3 \\ \vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_1 & |\vec{\mathbf{v}}_2|^2 & \vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_3 \\ \vec{\mathbf{v}}_3 \cdot \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_3 \cdot \vec{\mathbf{v}}_2 & |\vec{\mathbf{v}}_3|^2 \end{bmatrix} \qquad 5.1.6$$

and $\det T^\top T$ is given by

$$\begin{aligned} \det T^\top T &= |\vec{\mathbf{v}}_1|^2 |\vec{\mathbf{v}}_2|^2 |\vec{\mathbf{v}}_3|^2 + 2(\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2)(\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_3)(\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_3) \qquad 5.1.7 \\ &\quad - |\vec{\mathbf{v}}_1|^2 (\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_3)^2 - |\vec{\mathbf{v}}_2|^2 (\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_3)^2 - |\vec{\mathbf{v}}_3|^2 (\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2)^2 \\ &= |\vec{\mathbf{v}}_1|^2 |\vec{\mathbf{v}}_2|^2 |\vec{\mathbf{v}}_3|^2 \big(1 + 2\cos\theta_1 \cos\theta_2 \cos\theta_3 - (\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3)\big). \end{aligned}$$

For instance, the volume of a parallelepiped P spanned by three unit vectors, each making an angle of $\pi/4$ with the others, is

$$\mathrm{vol}_3\, P = \sqrt{1 + 2\cos^3\frac{\pi}{4} - 3\cos^2\frac{\pi}{4}} = \sqrt{\frac{\sqrt{2} - 1}{2}}. \qquad \triangle \qquad 5.1.8$$

It follows from Proposition 5.1.1 and equation 5.1.7 that we can express the volume of a n-parallelogram in $\mathbb{R}^n$ in terms of the lengths of its vectors and the angles between them: purely geometric information.

Equation 5.1.8: To use equation 5.1.1 to compute the volume a parallelepiped spanned by three unit vectors, each making an angle of $\pi/4$ with the others, we would first have to find appropriate vectors, which would not be easy.

Volume of a k-parallelogram in $\mathbb{R}^n$

The formula $\mathrm{vol}_k\, P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \sqrt{\det(T^\top T)}$ was useful in equation 5.1.8. But what really makes Proposition 5.1.1 interesting is that the same formula can be used to compute the area of a k-parallelogram in $\mathbb{R}^n$. Note that if T is an $n \times k$ matrix with columns $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$, then the product $T^\top T$ is a symmetric $k \times k$ matrix whose entries are dot products of the vectors $\vec{\mathbf{v}}_i$:

$$\overbrace{\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 & \dots & \vec{\mathbf{v}}_k \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}}^{T} \qquad 5.1.9$$

$$\underbrace{\begin{bmatrix} \dots & \vec{\mathbf{v}}_1^\top & \dots \\ \dots & \vec{\mathbf{v}}_2^\top & \dots \\ \dots & \dots & \dots \\ \dots & \vec{\mathbf{v}}_k^\top & \dots \end{bmatrix}}_{T^\top} \underbrace{\begin{bmatrix} |\vec{\mathbf{v}}_1|^2 & \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 & \dots & \vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_k \\ \vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_1 & |\vec{\mathbf{v}}_2|^2 & \dots & \vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_k \\ \vdots & \vdots & \ddots & \dots \\ \vec{\mathbf{v}}_k \cdot \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_k \cdot \vec{\mathbf{v}}_2 & \dots & |\vec{\mathbf{v}}_k|^2 \end{bmatrix}}_{T^\top T}.$$

There is another way to see that rotating the vectors $\vec{\mathbf{v}}_i$ by an orthogonal matrix A does not change $T^\top T$: the matrix T becomes the matrix AT, and

$$(AT)^\top AT = T^\top A^\top AT = T^\top T.$$

Exercise 5.1.6 asks you to use the singular value decomposition to give a different justification for using

$$\sqrt{\det T^\top T}$$

to define k-dimensional volume in $\mathbb{R}^n$.

Although T itself is not square, the matrix $T^\top T$ is square, and its entries can be computed from the lengths of the k vectors and the angles between them. No further information is needed. In particular, if the vectors are all rotated by the same orthogonal matrix, $T^\top T$ will be unchanged. Moreover, we do not need to know where the vectors are: at what point the parallelogram is anchored. Thus we can use $\sqrt{\det(T^\top T)}$ to define k-dimensional volume in $\mathbb{R}^n$.

Definition 5.1.3 (Volume of a k-parallelogram in $\mathbb{R}^n$). Let $T = [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]$ be an $n \times k$ real matrix. Then the *k-dimensional volume* of $P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is

$$\operatorname{vol}_k P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) \overset{\text{def}}{=} \sqrt{\det(T^\top T)}. \qquad 5.1.10$$

Exercise 5.1.5 asks you to show that $\det(T^\top T) \geq 0$, so that Definition 5.1.3 makes sense.

Example 5.1.4 (Volume of a 3-parallelogram in $\mathbb{R}^4$). Let P be the 3-parallelogram P in $\mathbb{R}^4$ spanned by $\vec{\mathbf{v}}_1 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$. What is its 3-dimensional volume? Set $T = [\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$; then

$$T^\top T = \begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{bmatrix} \quad \text{and} \quad \det(T^\top T) = 4, \quad \text{so } \operatorname{vol}_3 P = 2. \quad \triangle$$

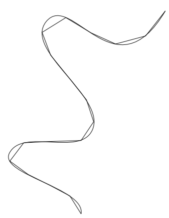

FIGURE 5.1.1.
A curve approximated by an inscribed polygon, shown already in Section 3.9.

Archimedes (287–212 BC) used this process to prove that

$$223/71 < \pi < 22/7.$$

In his famous paper *The Measurement of the Circle*, he approximated the circle by an inscribed and a circumscribed 96-sided regular polygon. That was the beginning of integral calculus.

Volume of anchored k-parallelograms

To break up a domain into little k-parallelograms we will need parallelograms "anchored" at different points in the domain. We denote by $P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ a k-parallelogram in $\mathbb{R}^n$ anchored at $\mathbf{x} \in \mathbb{R}^n$. Then

$$\operatorname{vol}_k P(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \operatorname{vol}_k P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k). \qquad 5.1.11$$

The anchored k-parallelograms are the "little pieces" we will use when breaking up the domain.

The "little number" assigned to each piece will be its volume.

The need for parametrizations

Now we must address a more complex issue. The first step in integration is to "break up the domain into little pieces". In Chapter 4 we had flat domains. Now we must break up a curvy domain into flat k-parallelograms.

For a curve, this is not hard. If $C \subset \mathbb{R}^n$ is a smooth curve, the integral $\int_C |d^1\mathbf{x}|$ is the number obtained by the following process: approximate C by little line segments as in Figure 5.1.1, apply $|d^1\mathbf{x}|$ to each to get its length, and add. Then let the approximation become infinitely fine; the limit is by definition the length of C.

It is much harder to define surface area. The obvious idea of taking the limit of the area of inscribed triangles as the triangles become smaller and smaller only works if we are careful to prevent the triangles from becoming skinny as they get small, and then it isn't obvious that such inscribed polyhedra exist at all (see Exercise 5.3.14).[1] The difficulties are not insurmountable, but they are daunting.

Instead we will base our definition of surface area (and, more generally, of k-dimensional volume of k-manifolds) on a parametrization: when

[1] We speak of triangles rather than parallelograms for the same reason that you would want a three-legged stool rather than a chair if your floor were uneven. You can make all three vertices of a triangle touch a curved surface; you can't do this for the four vertices of a parallelogram.