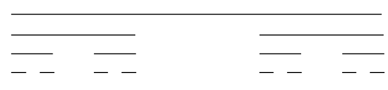
(If measuring length, $p=1$; if measuring area, $p=2$.) If the set had a length, the sum would converge when $p=1$, as $n\to\infty$; in fact, the sum is infinite. If it really had an area, then the power $p=2$ would lead to a finite limit; in fact, the sum is 0. But when $p=\ln 3/\ln 2$, the sum converges to $l^{\ln 3/\ln 2}\approx l^{1.58}$. This is the only dimension in which the Sierpinski gasket has finite, nonzero volume; in larger dimensions, the volume is 0; in smaller dimensions, it is infinite. $\triangle$

Exercises for Section 5.5

5.5.1 Consider the *triadic Cantor set* C obtained by removing from $[0,1]$ first the open middle third $(1/3,2/3)$, then the open middle third of each of the segments left, then the open middle third of each of the segments left, etc.:



a. Show that an alternative description of C is that it is the set of points that can be written in base 3 without using the digit 1. Use this to show that C is an uncountable set.

b. Show that C is a pavable set, with one-dimensional volume 0.

c. Show that the only dimension in which C can have volume different from 0 or infinity is $\ln 2/\ln 3$.

Hint for Exercise 5.5.1, part a: the number written as

$$.02220000022202002222\ldots$$

in base 3 is in C.

5.5.2 This time let the set C be obtained from the unit interval by omitting the open middle $1/n$th, then the open middle $1/n$th of each of the remaining intervals, then the open middle $1/n$th of the remaining intervals, etc. (When n is even, this means omitting an open interval equivalent to $1/n$th of the unit interval, leaving equal amounts on both sides, and so on.)

a. Show that C is a pavable set, with one-dimensional volume 0.

b. What is the only dimension in which C can have volume different from 0 or infinity? What is this dimension when $n=2$?

5.6 Review exercises for Chapter 5

5.1 Verify that equation 5.3.30 parametrizes the torus obtained by rotating around the z-axis the circle of radius r in the (x,z)-plane that is centered at $x=R,\ z=0$.

5.2 Let $f:[a,b]\to\mathbb{R}$ be a smooth positive function. Find a parametrization for the surface of equation $\dfrac{x^2}{A^2}+\dfrac{y^2}{B^2}=\big(f(z)\big)^2$.

5.3 For what values of α does the spiral $\begin{pmatrix} r(t)\\ \theta(t)\end{pmatrix}=\begin{pmatrix} 1/t^\alpha\\ t\end{pmatrix}$, $\alpha>0$ between $t=1$ and $t=\infty$ have finite length?

5.4 Compute the area of the graph of the function $f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{2}{3}(x^{3/2} + y^{3/2})$ above the region $0 \le x \le 1,\ 0 \le y \le 1$.

Exercise 5.5, part b: The answer should be in the form of a one-dimensional integral.

5.5 Let f be a positive C^1 function of $x \in [a, b]$.

a. Find a parametrization of the surface in $\mathbb{R}^3$ obtained by rotating the graph of f around the x-axis.

b. What is the area of this surface?

***5.6** Let

$$w_{n+1}(r) = \text{vol}_{n+1}(B_r^{n+1}(\mathbf{0}))$$

be the $(n+1)$-dimensional volume of the ball of radius r in $\mathbb{R}^{n+1}$, and let $\text{vol}_n(S_r^n)$ be the n-dimensional volume of the sphere of radius r in $\mathbb{R}^{n+1}$.

a. Show that $w'_{n+1}(r) = \text{vol}_n(S_r^n)$.

b. Show that $\text{vol}_n(S_r^n) = r^n \, \text{vol}_n(S_1^n)$.

c. Derive equation 5.3.50, using $w_{n+1}(1) = \int_0^1 w'_{n+1}(r)\, dr$.

The *total curvature* of a surface is defined in Exercise 5.3.11.

5.7 Let H be the helicoid of equation $y \cos z = x \sin z$ (see Example 3.9.13). What is the total curvature of the part of H with $0 \le z \le a$?

5.8 For $z \in \mathbb{C}$, the function $\cos z$ is by definition $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$.

a. If $z = x + iy$, write the real and imaginary parts of $\cos z$ in terms of x, y.

b. What is the area of the part of the graph of $\cos z$ where $-\pi \le x \le \pi$ and $-1 \le y \le 1$?

5.9 Let the set C be obtained from the unit interval $[0, 1]$ by omitting the open middle 1/5th, then the open middle fifth of each remaining interval, then the open middle fifth of each remaining interval, etc.

a. Show that an alternative description of C is that it is the set of points that can be written in base 5 without using the digit 2. Use this to show that C is an uncountable set.

b. Show that C is a pavable set, with one-dimensional volume 0.

c. What is the only dimension in which C can have volume different from 0 or infinity?

Exercise 5.10 is inspired by a proposition in *Functional Analysis, Volume 1: A Gentle Introduction*, by Dzung Minh Ha (Matrix Editions, 2006).

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{\mathbf{x}}_0}, \quad \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}}_{\vec{\mathbf{x}}_1}, \quad \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}}_{\vec{\mathbf{x}}_2}$$

Vectors for Exercise 5.10, part b

5.10 Let $\vec{\mathbf{x}}_0, \vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_k$ be vectors in $\mathbb{R}^n$, with $\vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_k$ linearly independent, and let $M \subset \mathbb{R}^n$ be the subspace spanned by $\vec{\mathbf{x}}_1, \dots, \vec{\mathbf{x}}_k$. Let G be the $k \times k$ matrix $G = [\vec{\mathbf{x}}_1 \dots \vec{\mathbf{x}}_k]^\top [\vec{\mathbf{x}}_1 \dots \vec{\mathbf{x}}_k]$ and let G^+ be the $(k+1) \times (k+1)$ matrix

$$G^+ = [\vec{\mathbf{x}}_0\, \vec{\mathbf{x}}_1 \dots \vec{\mathbf{x}}_k]^\top [\vec{\mathbf{x}}_0\, \vec{\mathbf{x}}_1 \dots \vec{\mathbf{x}}_k].$$

The *distance* $d(\vec{\mathbf{x}}, M)$ is by definition $d(\vec{\mathbf{x}}, M) = \inf_{\vec{\mathbf{y}} \in M} (\vec{\mathbf{x}} - \vec{\mathbf{y}})$.

a. Show that

$$\Big(d(\vec{\mathbf{x}}_0, M)\Big)^2 = \frac{\det G^+}{\det G}.$$

b. What is the distance between $\vec{\mathbf{x}}_0$ and the plane M spanned by $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ (as defined in the margin)?