4.12 Review exercises for Chapter 4

Exercise 4.1:

$U_N(\mathbf{1}_C) = L_N(\mathbf{1}_C)$.

4.1 Show that if $C \in \mathcal{D}(\mathbb{R}^m)$, then $\mathbf{1}_C$ is integrable.

4.2 An integrand should take a piece of the domain and return a number, in such a way that if we decompose a domain into little pieces, evaluate the integrand on the pieces, and add, the sums should have a limit as the decomposition becomes infinitely fine (and the limit should not depend on how the domain is decomposed). What will happen if we break up $[0,1]^2$ into rectangles defined by $a < x < b$ and $c < y < d$ and assign one of the numbers below to each rectangle?

a. $|ac - bd|$ b. $(ad - bc)^2$.

4.3 Let A be an $n \times n$ matrix of integers, viewed as a map $\mathbb{Z}^n \to \mathbb{Z}^n$. Which of the following are true?

1. $\ker A = 0 \implies A$ is onto.
2. A onto $\implies \ker A = 0$.
3. $\det A \neq 0 \implies \ker A = 0$.
4. $\det A \neq 0 \implies A$ is onto.

4.4 Which elementary matrices are permutation matrices? Describe the corresponding permutations.

4.5 Evaluate $\displaystyle\lim_{N\to\infty} \frac{1}{N^2} \sum_{k=1}^{N} \sum_{l=1}^{2N} e^{\frac{k+l}{N}}$.

4.6 What are the expectation, variance, and standard deviation, if they exist, of the random variable $f(x) = x$, for the following probability densities.

a. $\mu(x) = e^{-x}\mathbf{1}_{[0,\infty]}$ b. $\mu(x) = \dfrac{1}{(x+1)^2}\mathbf{1}_{[0,\infty)}$ c. $\mu(x) = \dfrac{2}{(x+1)^3}\mathbf{1}_{[0,\infty)}$

4.7 Let A and B be two disjoint bodies, with densities μ_1 and μ_2 and masses $M(A)$ and $M(B)$. Set $C = A \cup B$. Show that the center of gravity of C is

$$\overline{\mathbf{x}}(C) = \frac{M(A)\overline{\mathbf{x}}(A) + M(B)\overline{\mathbf{x}}(B)}{M(A) + M(B)}.$$

4.8 Choose r and R with $0 < r < R < \infty$.

a. Find the integral

$$\int_{A_{r,R}} \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}} \quad \text{over the region } r^2 \leq x^2+y^2+z^2 \leq R^2.$$

b. Does this integral have a limit as $R \to \infty$? As $r \to 0$?

4.9 Let X be a subset of $\mathbb{R}^n$ such that for any $\epsilon > 0$, there exists a sequence $i \mapsto B_i$ of pavable sets satisfying $X \subset \bigcup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} \mathrm{vol}_n(B_i) < \epsilon$. Show that X has measure 0.

4.10 Give an explicit upper bound (in terms of N) for the number of cubes in $\mathcal{D}_N(\mathbb{R}^3)$ needed to cover the unit sphere $S^2 \subset \mathbb{R}^3$, such that the volume of the cubes tends to 0 as N tends to infinity.

4.11 Write each of the following double integrals as iterated integrals in two ways, and compute them:

a. The integral of $\sin(x+y)$ over the region $x^2 < y < 2$

b. The integral of $x^2 + y^2$ over the region where $1 \leq |x| \leq 2$ and $1 \leq |y| \leq 2$

4.12 Compute the integral of the function z over the region R described by the inequalities $x > 0,\ y > 0,\ z > 0,\ x + 2y + 3z < 1$.

4.13 a. If $f\begin{pmatrix} x \\ y \end{pmatrix} = a + bx + cy$, what are

$$\int_0^1 \int_0^2 f\begin{pmatrix} x \\ y \end{pmatrix} |dx\, dy| \quad \text{and} \quad \int_0^1 \int_0^2 \left(f\begin{pmatrix} x \\ y \end{pmatrix}\right)^2 |dx\, dy|?$$

b. Let f be as in part a. What is the minimum of $\int_0^1 \int_0^2 \left(f\begin{pmatrix} x \\ y \end{pmatrix}\right)^2 |dx\, dy|$ among all functions f such that

$$\int_0^1 \int_0^2 f\begin{pmatrix} x \\ y \end{pmatrix} |dx\, dy| = 1?$$

4.14 What is the z-coordinate of the center of gravity of the region

$$\frac{x^2}{(z^3-1)^2} + \frac{y^2}{(z^3+1)^2} \le 1, \quad 0 \le z \le 1\,?$$

4.15 Show that there exist c and u such that when f is any polynomial of degree $d \le 3$,

$$\int_{-1}^{1} f(x) \frac{1}{\sqrt{1-x^2}}\, dx = c\Big(f(u) + f(-u)\Big).$$

4.16 Repeat Exercise 4.6.4, parts a–d, but this time for the weight e^{-x^2} and limits of integration $-\infty$ to ∞; i.e., find points x_i and w_i such that

$$\int_{-\infty}^{\infty} p(x) e^{-x^2}\, dx = \sum_{i=0}^{k} w_i p(x_i)$$

for all polynomials of degree $\le 2k-1$.

e. For each of the four values of m in part d, approximate

$$\int_{-\infty}^{\infty} e^{-x^2} \cos x\, dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2}\, dx.$$

Compare the approximations with the exact values.

4.17 Check part 3 of Theorem 4.8.15 when $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; that is, show that $[\mathbf{D} \det(A)]B = \det A \ \mathrm{tr}(A^{-1}B)$.

4.18 Show that if A and B are $n \times n$ matrices, then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Exercise 4.18: Start with Corollary 4.8.17, and set

$$C = P \text{ and } D = AP^{-1}.$$

This proves the formula when C is invertible. Complete the proof by showing that if $n \mapsto C_n$ is a sequence of matrices converging to C, and $\mathrm{tr}(C_n D) = \mathrm{tr}(DC_n)$ for all n, then $\mathrm{tr}(CD) = \mathrm{tr}(DC)$.

4.19 What is the n-dimensional volume of the region

$$\{\, \mathbf{x} \in \mathbb{R}^n \mid x_i \ge 0 \text{ for all } i = 1, \dots, n \text{ and } x_1 + 2x_2 + \cdots + nx_n \le n \,\}?$$

4.20 a. Find an expression for the area of the parallelogram spanned by $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, in terms of $|\vec{\mathbf{v}}_1|, |\vec{\mathbf{v}}_2|$, and $|\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2|$.

b. Prove Heron's formula: A triangle with sides of length a, b, c, has area

$$\sqrt{p\,(p-a)(p-b)(p-c)}\ , \quad \text{where} \quad p = \frac{a+b+c}{2}.$$

4.21 a. Sketch the curve in the plane given in polar coordinates by the equation $r = 1 + \sin\theta, \quad 0 \le \theta \le 2\pi$.

b. Find the area that the curve encloses.

4.22 A semicircle of radius R has density $\rho\begin{pmatrix} x \\ y \end{pmatrix} = m(x^2+y^2)$ proportional to the square of the distance to the center. What is its mass?

4.23 a. Let Q be the part of the unit ball $x^2+y^2+z^2 \le 1$ where $x, y, z \ge 0$. Using spherical coordinates, set up $\int_Q (x+y+z)\, |d^3\mathbf{x}|$ as an iterated integral.

b. Compute the integral.

4.24 Let A be a square matrix. Show that A and $A^\top$ have the same eigenvalues, with the same multiplicities.

4.25 Let $A \subset \mathbb{R}^3$ be the region defined by the inequalities $x^2+y^2 \le z \le 1$. What is the center of gravity of A?

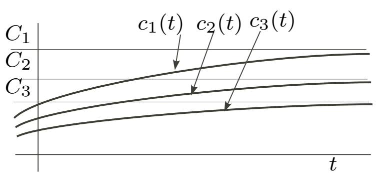

FIGURE FOR EXERCISE 4.26
The series $n \mapsto c_n(t)$ for part d. The functions $c_n(t)$ are positive monotone increasing as functions of t, and decreasing as functions of n.

4.26 In this exercise we will show that $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$. This function is not Lebesgue integrable, and the integral should be understood as

$$\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{a\to\infty} \int_0^a \frac{\sin x}{x}\, dx.$$

a. Show that for all $0 < a < b < \infty$,

$$\int_a^b \left(\int_0^\infty e^{-px} \sin x\, dx\right) dp = \int_0^\infty \left(\int_a^b e^{-px} \sin x\, dp\right) dx.$$

b. Use part a to show

$$\arctan b - \arctan a = \int_0^\infty \frac{(e^{-ax} - e^{-bx}) \sin x}{x}\, dx.$$

c. Why does Theorem 4.11.4 not imply that

$$\lim_{a\to 0} \lim_{b\to\infty} \int_0^\infty \frac{(e^{-ax} - e^{-bx}) \sin x}{x}\, dx = \int_0^\infty \frac{\sin x}{x}\, dx? \qquad (1)$$

Exercise 4.26, part d: Remember that the next omitted term is a bound for the error for each partial sum.

d. Prove that equation 1 in part c is true anyway. The following lemma is the key: if $n \mapsto c_n(t)$ is a sequence of positive monotone increasing functions of t, with $\lim_{t\to\infty} c_n(t) = C_n$, and decreasing as a function of n for each fixed t, tending to 0 (see the figure in the margin), then

$$\lim_{t\to\infty} \sum_{n=1}^\infty (-1)^n c_n(t) = \sum_{n=1}^\infty (-1)^n C_n.$$

e. Write

$$\int_0^\infty \frac{(e^{-ax} - e^{-bx}) \sin x}{x}\, dx = \sum_{n=0}^\infty \int_{k\pi}^{(k+1)\pi} (-1)^k \frac{(e^{-ax} - e^{-bx})|\sin x|}{x}\, dx,$$

and use part d to prove the equation $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

$$f(x) = \sum_{k=1}^\infty \frac{1}{2^k} \frac{1}{\sqrt{|x - a_k|}}.$$

Function f for Exercise 4.27

Exercise 4.27, part c: This depends on the order chosen.

4.27 Let $a_1, a_2, \ldots$ be a list of the rationals in $[0, 1]$.

a. Show that the function f given in the margin is L-integrable on $[0, 1]$.

b. Show that the series converges for all x except x on a set of measure 0.

*c. Find an x for which the series converges.

4.28 a. Show that $\frac{1}{x^2} \mathbf{1}_{[1,\infty]}(x)$ is a probability density.

b. Show that for the probability density found in part a, the random variable $f(x) = x$ does not have an expectation (i.e., that the expectation is infinite).

c. Show that $\frac{2}{x^3}\mathbf{1}_{[1,\infty]}(x)$ is a probability density.

d. Show that the random variable $f(x) = x$ has an expectation with respect to the probability density of part c; compute it. Show that it does not have a variance (i.e., the variance is infinite).

4.29 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(u) = au + b\bar{u}$, where we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the standard way. Show that

$$\det T = |a|^2 - |b|^2 \quad \text{and} \quad \|T\| = |a| + |b|.$$

Exercise 4.30: Think of Riemann sums.

4.30 a. Find the unique integer p such that the limit

$$\lim_{n\to\infty} \frac{1}{n^p} \sum_{\substack{k,l,m \text{ integers} \\ 0\le k,\, 0\le l,\, 0\le m \\ k^2+l^2+m^2\le n^2}} \frac{klm}{k^2+l^2+m^2}$$

exists and is nonzero. For that value of p, compute the limit.

b. What happens to the limit if we replace p by p' with $p' < p$? with $p' > p$?

4.31 For $a > 1,\ b > 1,\ c > 1$, what is the volume of the part of the region

$$A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 1 \le xyz \le a,\ 1 \le x \le b,\ xz \le y \le cxz \right\}$$

where $x > 0, y > 0, z > 0$?

a. Find a change of variables, expressing appropriate $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ in terms of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, so that A becomes a parallelopiped in the new variables.

b. Invert the change of variables, and find its derivative and the determinant of the derivative.

c. Compute the integral.

$$\int_0^1 \left(\int_y^{y^{1/3}} \sin(x^2)\, dx \right) dy$$

Integral for Exercise 4.32.

4.32 a. Transform the iterated integral in the margin into an integral over a subset of $\mathbb{R}^2$. Sketch this subset.

b. Compute the integral.

***4.33** Find an open subset $U \subset [0,1]$ and a continuous function $f : U \to [0,1]$ whose graph $\Gamma_f \subset [0,1] \times [0,1]$ does not have volume. Show that it has 2-dimensional measure 0. *Hint*: Look at Remark 5.2.5.

4.34 Let $i \mapsto a_i$ be a list of the rationals in $[0,1]$, and set

$$f_k(x) = \begin{cases} 1 & \text{if } x \in [0,1] \text{ and } |x - a_i| < \frac{1}{10^i} \text{ for } i \le k \\ 0 & \text{otherwise.} \end{cases}$$

Show that the f_k are Riemann integrable but that they cannot be modified on a set of measure 0 to converge to a Riemann-integrable function.

4.35 Let $\mathbf{x} \in \mathbb{R}^n$. For what values of $p \in \mathbb{R}$ does $\int_{B_1(\mathbf{0})} |\mathbf{x}|^p \,|d^n\mathbf{x}|$ exist as a Lebesgue integral? (The answer depends on n.)