b. Show that $\Phi : Q_a \to A$ is one to one.

c. What is $\int_A y\,|dx\,dy|$?

4.10.18 What is the volume of the part of the ball of equation $x^2+y^2+z^2 \le 4$ where $z^2 \ge x^2+y^2,\ z>0$?

4.10.19 Let $Q = [0,1]\times[0,1]$ be the unit square in $\mathbb{R}^2$, and let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - v^2 \\ u^2 + v \end{pmatrix}. \quad \text{Set } A = \Phi(Q).$$

a. Sketch A, by computing the image of each of the sides of Q (they are all arcs of parabolas).

b. Show that $\Phi : Q \to A$ is 1–1.

c. What is $\int_A x\,|dx\,dy|$?

Change of variables for Exercise 4.10.20:

$$\Phi\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} (w^3-1)u \\ (w^3+1)v \\ w \end{pmatrix}.$$

4.10.20 Solve Exercise 4.5.19 again, using the change of variables in the margin.

4.10.21 The *moment of inertia* of a body $X \subset \mathbb{R}^3$ around an axis is the integral $\int_X (r(\mathbf{x}))^2 |d^3\mathbf{x}|$, where $r(\mathbf{x})$ is the distance from $\mathbf{x}$ to the axis.

a. Let f be a nonnegative continuous function of $x \in [a,b]$, and let B be the body obtained by rotating the region $0 \le y \le f(x),\ a \le x \le b$ around the x-axis. What is the moment of inertia of B around the x-axis?

b. What number does this give when $f(x) = \cos x, \quad a = -\frac{\pi}{2}, \quad b = \frac{\pi}{2}$?

4.11 Lebesgue integrals

This new integral of Lebesgue is proving itself a wonderful tool. I might compare it with a modern Krupp gun, so easily does it penetrate barriers which were impregnable.—Edward Van Vleck, *Bulletin of the American Mathematical Society*, vol. 23, 1916.

So far we have restricted ourselves to integrals of bounded functions with bounded support, whose upper and lower sums are equal. But often we will want to integrate functions that are not bounded or do not have bounded support. Lebesgue integration makes this possible. It also has two other advantages:

There are many reasons to study Lebesgue integrals. An essential one is the Fourier transform, the fundamental tool of engineering and signal processing, not to mention harmonic analysis. (The Fourier transform is discussed at the end of this section.) Lebesgue integrals are also ubiquitous in probability theory.

1. Lebesgue integrals exist for functions plagued with the kind of "local nonsense" that we saw in the function that is 1 at rational numbers in $[0,1]$ and 0 elsewhere (Example 4.3.3). The Lebesgue integral ignores local nonsense on sets of measure 0.
2. Lebesgue integrals are better behaved with respect to limits.

Remark. If Lebesgue integration is superior to Riemann integration, why did we put so much emphasis on Riemann integration earlier in this chapter? Riemann integrals have one great advantage over Lebesgue integrals: they can be computed using Riemann sums. Lebesgue integrals can only be computed via Riemann integrals (or perhaps by using Monte Carlo methods). Thus our approach is in keeping with our emphasis on computationally effective algorithms. △

Before defining the Lebesgue integral, we will discuss the behavior of Riemann integrals with respect to limits.

Our approach to Lebesgue integration is very different from the standard one. The usual way of defining the Lebesgue integral

$$\int_{\mathbb{R}^n} f(\mathbf{x})|d^n\mathbf{x}|$$

is to cut up the *codomain* $\mathbb{R}$ into small intervals $I_i = [x_i, x_{i+1}]$, and to approximate the integral by

$$\sum_i x_i \mu(f^{-1}(I_i)),$$

where $\mu(A)$ is the *measure of* A, then letting the decomposition of the codomain become arbitrarily fine. Of course, this requires saying what subsets are measurable, and defining their measure. This is the main task with the standard approach, and for this reason the theory of Lebesgue integration is often called *measure theory*.

It is surprising how much more powerful the theory is when one decomposes the codomain rather than the domain. But one pays a price: it isn't at all clear how one would approximate a Lebesgue integral: figuring out what the sets $f^{-1}(I_i)$ are, never mind finding their measure, is difficult or impossible even for the simplest functions.

We take a different tack, building on the theory of Riemann integrals, and defining the integral directly by taking limits of functions that are Riemann integrable. We get measure theory at the end as a byproduct: just as Riemann integrals are used to define volume, Lebesgue integrals can be used to define measure. This is discussed in Appendix A21.

Integrals and limits

The behavior of integrals under limits is often important. Here we give the best general statements about Riemann integrals and limits.

We would like to be able to say that if $k \mapsto f_k$ is a convergent sequence of functions, then, as $k \to \infty$,

$$\int \lim f_k = \lim \int f_k. \qquad 4.11.1$$

In one setting this is true and straightforward: when $k \mapsto f_k$ is a *uniformly convergent* sequence of integrable functions, all with support in the same bounded set. The key condition in Definition 4.11.1 is that given ϵ, the same K works for all $\mathbf{x}$.

Definition 4.11.1 (Uniform convergence). A sequence $k \mapsto f_k$ of functions $f_k : \mathbb{R}^n \to \mathbb{R}$ *converges uniformly* to a function f if for every $\epsilon > 0$, there exists K such that when $k \geq K$, then, for all $\mathbf{x} \in \mathbb{R}^n$, $|f_k(\mathbf{x}) - f(\mathbf{x})| < \epsilon$.

The three sequences of functions in Example 4.11.3 do *not* converge uniformly, although they do converge. Uniform convergence on all of $\mathbb{R}^n$ isn't a very common phenomenon, unless something is done to cut down the domain. For instance, suppose that $k \mapsto p_k$ is a sequence of polynomials

$$p_k(x) = a_{0,k} + a_{1,k}x + \cdots + a_{m,k}x^m \qquad 4.11.2$$

all of degree $\leq m$, and that this sequence "converges" in the "obvious" sense that for each degree i (i.e., each x^i), the sequence of coefficients $a_{i,0}, a_{i,1}, a_{i,2}, \ldots$ converges. Then $k \mapsto p_k$ does not converge uniformly on $\mathbb{R}$. But for any bounded set A, the sequence $k \mapsto p_k\mathbf{1}_A$ does converge uniformly.[15]

[15]Instead of writing $p_k\mathbf{1}_A$ we could write "p_k restricted to A".