3

Manifolds, Taylor polynomials, quadratic forms, and curvature

Thomson [Lord Kelvin] had predicted the problems of the first [transatlantic] cable by mathematics. On the basis of the same mathematics he now promised the company a rate of eight or even 12 words a minute. Half a million pounds was being staked on the correctness of a partial differential equation.—T. W. Körner, *Fourier Analysis*

3.0 Introduction

This chapter is something of a grab bag. The various themes are related, but the relationship is not immediately apparent. We begin with two sections on geometry. In Section 3.1 we use the implicit function theorem to define smooth curves, smooth surfaces, and more general k-dimensional "surfaces" in $\mathbb{R}^n$, called *manifolds*. In Section 3.2 we discuss linear approximations to manifolds: *tangent spaces*.

We switch gears in Section 3.3, where we use higher partial derivatives to construct the Taylor polynomial of a function in several variables. We saw in Section 1.7 how to approximate a nonlinear function by its derivative; here we see that we can better approximate C^k functions using Taylor polynomials when $k \geq 2$. This is useful, since polynomials, unlike sines, cosines, exponentials, square roots, logarithms, ... can actually be computed using arithmetic. Computing Taylor polynomials by calculating higher partial derivatives can be quite unpleasant; in Section 3.4 we show how to compute them by combining Taylor polynomials of simpler functions.

In Sections 3.5 and 3.6 we take a brief detour, introducing quadratic forms and seeing how to classify them according to their "signature": if we consider the second-degree terms of a function's Taylor polynomial as a quadratic form, its signature usually tells us whether at a point where the derivative vanishes the function has a minimum value, a maximum value, or some kind of *saddle*, like a mountain pass. In Section 3.7 we use Lagrange multipliers to find extrema of a function restricted to some manifold $M \subset \mathbb{R}^n$; we use Lagrange multipliers to prove the *spectral theorem*.

In Section 3.8 we introduce finite probability spaces, and show how the singular value decomposition (a consequence of the spectral theorem) gives rise to *principal component analysis*, of immense importance in statistics.

In Section 3.9 we give a brief introduction to the vast and important subject of the geometry of curves and surfaces, using the higher-degree approximations provided by Taylor polynomials: the *curvature* of a curve or surface depends on the quadratic terms of the functions defining it, and the *torsion* of a space curve depends on the cubic terms.

When a computer calculates sines, it does not look up the answer in some mammoth table of sines; stored in the computer is a polynomial that very well approximates $\sin x$ for x in some particular range. Specifically, it uses a formula very close to equation 3.4.6:

$$\sin x = x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + a_{11}x^{11} + \epsilon(x),$$

where the coefficients are

$$\begin{aligned} a_3 &= -.1666666664 \\ a_5 &= .0083333315 \\ a_7 &= -.0001984090 \\ a_9 &= .0000027526 \\ a_{11} &= -.0000000239. \end{aligned}$$

When

$$|x| \leq \pi/2,$$

the error $\epsilon(x)$ is guaranteed to be less than 2×10^{-9}, good enough for a calculator that computes to eight significant digits. The computer needs only to remember five coefficients and do a bit of arithmetic to replace a huge table of sines.

3.1 Manifolds

Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions—Felix Klein

In this section we introduce one more actor in multivariable calculus. So far, our mappings have been first linear, then nonlinear with good linear approximations. But the domain and codomain of our mappings have been "flat" open subsets of $\mathbb{R}^n$. Now we want to allow "nonlinear" $\mathbb{R}^n$'s, called *smooth manifolds*.

These familiar objects are by no means simple: already, the theory of soap bubbles is a difficult topic, with a complicated partial differential equation controlling the shape of the film.

Manifolds are a generalization of the familiar curves and surfaces of every day experience. A one-dimensional manifold is a smooth curve; a two-dimensional manifold is a smooth surface. Smooth curves are idealizations of things like telephone wires or a tangled garden hose. Particularly beautiful smooth surfaces are produced when you blow soap bubbles that wobble and slowly vibrate as they drift through the air. Other examples are shown in Figure 3.1.2.

Figure 3.1.1.
Felix Klein (1849–1925)

Klein's work in geometry "has become so much a part of our present mathematical thinking that it is hard for us to realise the novelty of his results."—From a biographical sketch by J. O'Connor and E. F. Robertson. Klein was also instrumental in developing *Mathematische Annalen* into one of the most prestigious mathematical journals.

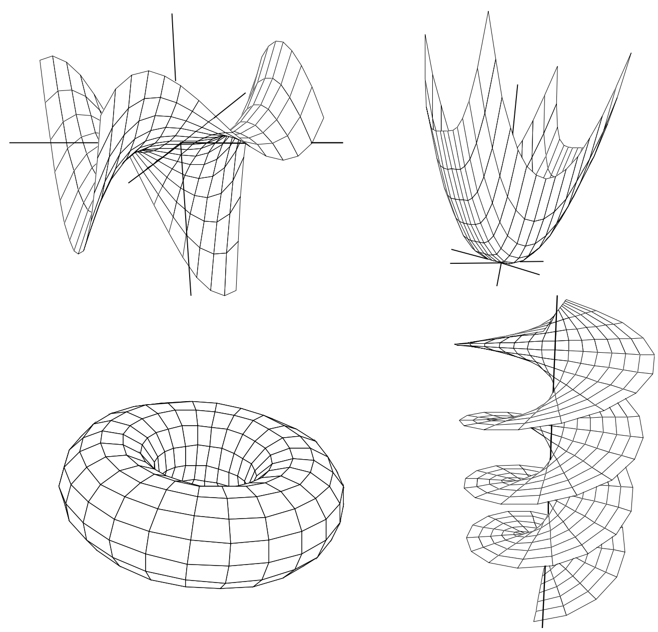

Figure 3.1.2. Four surfaces in $\mathbb{R}^3$. The top two are graphs of functions. The bottom two are *locally* graphs of functions. All four qualify as smooth surfaces (two-dimensional manifolds) under Definition 3.1.2.

We will define smooth manifolds mathematically, excluding some objects that we might think of as smooth: a figure eight, for example. We will see how to use the implicit function theorem to tell whether the locus defined by an equation is a smooth manifold. Finally, we will compare knowing a manifold by equations, and knowing it by a parametrization.

Smooth manifolds in $\mathbb{R}^n$

When is a subset $X \subset \mathbb{R}^n$ a smooth manifold? Our definition is based on the notion of a graph.

It is convenient to denote a point in the graph of a function $\mathbf{f}: \mathbb{R}^k \to \mathbb{R}^{n-k}$ as $\begin{pmatrix} \mathbf{x} \\ \mathbf{f}(\mathbf{x}) \end{pmatrix}$, with $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^{n-k}$. But this presupposes that the "active" variables are the first variables, which is a problem, since we usually cannot use the same active variables at all points of the manifold.

How then might we describe a point $\mathbf{z}$ in the graph of a function $\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^{n-k}$, where the variables of the domain of $\mathbf{f}$ have indices $i_1, \dots, i_k$ and the variables of the codomain have indices $j_1, \dots, j_{n-k}$? Here is an accurate if heavy-handed approach, using the "concrete to abstract" linear transformation Φ of Definition 2.6.12. Set

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-k} \end{pmatrix}$$

with $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Define Φ_{d} (d for the domain of $\mathbf{f}$) and Φ_{c} (c for codomain) by

$$\Phi_{\mathrm{d}}(\mathbf{x}) = x_1 \vec{\mathbf{e}}_{i_1} + \cdots + x_k \vec{\mathbf{e}}_{i_k}$$
$$\Phi_{\mathrm{c}}(\mathbf{y}) = y_1 \vec{\mathbf{e}}_{j_1} + \cdots + y_{n-k} \vec{\mathbf{e}}_{j_{n-k}},$$

where the $\vec{\mathbf{e}}$'s are standard basis vectors in $\mathbb{R}^n$. Then the graph of $\mathbf{f}$ is the set of $\mathbf{z}$ such that

$$\mathbf{z} = \Phi_{\mathrm{d}}(\mathbf{x}) + \Phi_{\mathrm{c}}(\mathbf{f}(\mathbf{x})).$$

Since the function $\mathbf{f}$ of Definition 3.1.2 is C^1, its domain must be open. If $\mathbf{f}$ is C^p rather than C^1, then the manifold is a C^p manifold.

Definition 3.1.1 (Graph). The *graph* of a function $\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is the set of points $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^n$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. It is denoted $\Gamma(\mathbf{f})$.

Thus the graph of a function $\mathbf{f}$ lives in a space whose dimension is the sum of the dimensions of the domain and codomain of $\mathbf{f}$.

Traditionally we graph functions $f : \mathbb{R} \to \mathbb{R}$ with the horizontal x-axis corresponding to the input, and the vertical axis corresponding to the output (values of f). Note that the graph of such a function is a subset of $\mathbb{R}^2$. For example, the graph of $f(x) = x^2$ consists of the points $\begin{pmatrix} x \\ f(x) \end{pmatrix} \in \mathbb{R}^2$, i.e., the points $\begin{pmatrix} x \\ x^2 \end{pmatrix}$.

The top two surfaces shown in Figure 3.1.2 are graphs of functions from $\mathbb{R}^2$ to $\mathbb{R}$: the surface on the left is the graph of $f\begin{pmatrix} x \\ y \end{pmatrix} = x^3 - 2xy^2$; that on the right is the graph of $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^4$. Although we depict these graphs on a flat piece of paper, they are actually subsets of $\mathbb{R}^3$. The first consists of the points $\begin{pmatrix} x \\ y \\ x^3 - 2xy^2 \end{pmatrix}$, the second of the points $\begin{pmatrix} x \\ y \\ x^2 + y^4 \end{pmatrix}$.

Definition 3.1.2 says that if a function $\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is C^1, then its graph is a smooth n-dimensional manifold in $\mathbb{R}^n$. Thus the top two graphs shown in Figure 3.1.2 are two-dimensional manifolds in $\mathbb{R}^3$.

But the torus and helicoid shown in Figure 3.1.2 are also two-dimensional manifolds. Neither is the graph of a single function expressing one variable in terms of the other two. But both are *locally* graphs of functions.

Definition 3.1.2 (Smooth manifold in $\mathbb{R}^n$). A subset $M \subset \mathbb{R}^n$ is a smooth k-dimensional manifold if locally it is the graph of a C^1 mapping $\mathbf{f}$ expressing $n - k$ variables as functions of the other k variables.

With this definition, which depends on chosen coordinates, it isn't obvious that if you rotate a smooth manifold it is still smooth. We will see in Theorem 3.1.16 that it is.

Generally, "smooth" means "as many times differentiable as is relevant to the problem at hand". In this and the next section, it means of class C^1. When speaking of smooth manifolds, we often omit the word smooth.[1]

"Locally" means that every point $\mathbf{x} \in M$ has a neighborhood $U \subset \mathbb{R}^n$ such that $M \cap U$ (the part of M in U) is the graph of a mapping expressing

[1]Some authors use "smooth" to mean C^∞: "infinitely many times differentiable". For our purposes this is overkill.

$n-k$ of the coordinates of each point in $M \cap U$ in terms of the other k. This may sound like an unwelcome complication, *but if we omitted the word "locally" then we would exclude from our definition most interesting manifolds.* We already saw that neither the torus nor the helicoid of Figure 3.1.2 is the graph of a single function expressing one variable as a function of the other two. Even such a simple curve as the unit circle is not the graph of a single function expressing one variable in terms of the other. In Figure 3.1.3 we show another smooth curve that would not qualify as a manifold if we required it to be the graph of a single function expressing one variable in terms of the other; the caption justifies our claim that this curve is a smooth curve.

A manifold M embedded in $\mathbb{R}^n$, denoted $M \subset \mathbb{R}^n$, is sometimes called a *submanifold* of $\mathbb{R}^n$. Strictly speaking, it should not be referred to simply as a "manifold", which can mean an abstract manifold, not embedded in any space. The manifolds in this book are all submanifolds of $\mathbb{R}^n$ for some n.

Especially in higher dimensions, making some kind of global sense of a patchwork of graphs of functions can be quite challenging; a mathematician trying to picture a manifold is rather like a blindfolded person who has never met or seen a picture of an elephant seeking to identify one by patting first an ear, then the trunk or a leg. It is a subject full of open questions, some fully as interesting and demanding as, for example, Fermat's last theorem, whose solution after more than three centuries aroused such passionate interest.

Three-dimensional and four-dimensional manifolds are of particular interest, in part because of applications in representing space-time.

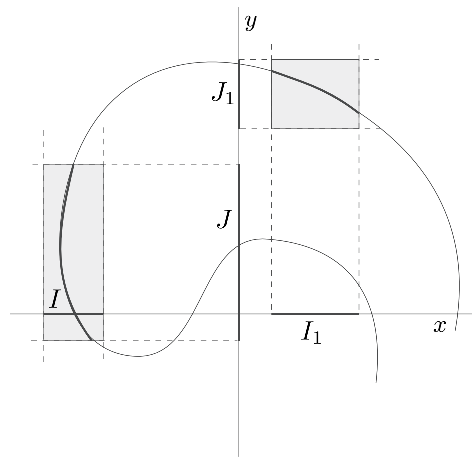

FIGURE 3.1.3. Above, I and I_1 are intervals on the x-axis; J and J_1 are intervals on the y-axis. The darkened part of the curve in the shaded rectangle $I \times J$ is the graph of a function expressing $x \in I$ as a function of $y \in J$, and the darkened part of the curve in $I_1 \times J_1$ is the graph of a function expressing $y \in J_1$ as a function of $x \in I_1$. (By decreasing the size of J_1 a bit, we could also think of the part of the curve in $I_1 \times J_1$ as the graph of a function expressing $x \in I_1$ as a function of $y \in J_1$.) But we cannot think of the darkened part of the curve in $I \times J$ as the graph of a function expressing $y \in J$ as a function of $x \in I$; there are values of x that give two different values of y, and others that give none, so such a "function" is not well defined.

Example 3.1.3 (Graph of smooth function is smooth manifold). The graph of any smooth function is a smooth manifold. The curve of equation $y = x^2$ is a one-dimensional manifold: the graph of y as the function $f(x) = x^2$. The curve of equation $x = y^2$ is also a one-dimensional manifold: the graph of a function representing x as a function of y. Each surface at the top of Figure 3.1.2 is the graph of a function representing z as a function of x and y. △

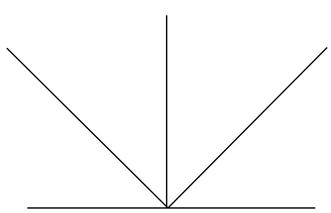

FIGURE 3.1.4.

The graph of $f(x) = |x|$ is not a smooth curve.

Example 3.1.4 (Graphs that are not smooth manifolds). The graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$, shown in Figure 3.1.4, is not a