2.10.14 Consider the mapping $S : \text{Mat}\,(2,2) \to \text{Mat}\,(2,2)$ given by $S(A) = A^2$. Observe that $S(-I) = I$. Does there exist an inverse mapping g, i.e., a mapping such that $S(g(A)) = A$, defined in a neighborhood of I, and such that $g(I) = -I$?

2.10.15 a. Show that the mapping $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^x + e^y \\ e^x + e^{-y} \end{pmatrix}$ is locally invertible at every point $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

b. If $F(\mathbf{a}) = \mathbf{b}$, what is the derivative of F^{-1} at $\mathbf{b}$?

2.10.16 The matrix $A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ satisfies $A_0^3 = I$. True or false? There exists a neighborhood $U \subset \text{Mat}\,(3,3)$ of I and a continuously differentiable function $g : U \to \text{Mat}\,(3,3)$ with $g(I) = A_0$ and $(g(A))^3 = A$ for all $A \in U$ (i.e., $g(A)$ is a cube root of A).

2.10.17 Prove Theorem 2.10.2 (the inverse function theorem in one dimension).

2.11 Review exercises for Chapter 2

$$x + y - z = a$$
$$x + 2z = b$$
$$x + ay + z = b$$

Equations for Exercise 2.1

2.1 a. For what values of a and b does the system of linear equations shown in the margin have one solution? No solutions? Infinitely many solutions?

b. For what values of a and b is the matrix of coefficients invertible?

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

Matrix A of Exercise 2.2

2.2 When A is the matrix at left, multiplication by what elementary matrix corresponds to

a. Exchanging the first and second rows of A?

b. Multiplying the fourth row of A by 3?

c. Adding 2 times the third row of A to the first row of A?

2.3 a. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Are the following statements true or false?

1. If $\ker T = \{\vec{\mathbf{0}}\}$ and $T(\vec{\mathbf{y}}) = \vec{\mathbf{b}}$, then $\vec{\mathbf{y}}$ is the only solution to $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$.
2. If $\vec{\mathbf{y}}$ is the only solution to $T(\vec{\mathbf{x}}) = \vec{\mathbf{c}}$, then for any $\vec{\mathbf{b}} \in \mathbb{R}^m$, a solution exists to $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$.
3. If $\vec{\mathbf{y}} \in \mathbb{R}^n$ is a solution to $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$, it is the only solution.
4. If for any $\vec{\mathbf{b}} \in \mathbb{R}^m$ the equation $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$ has a solution, then it is the only solution.

b. For any statements that are false, can one impose conditions on m and n that make them true?

2.4 a. Show that an upper triangular matrix is invertible if and only if its diagonal entries are all nonzero, and that if it is invertible, its inverse is upper triangular.

b. Show the corresponding statement for a lower triangular matrix.

2.5 a. Let A be an $(n-k)\times(n-k)$ matrix, B an $k\times k$ matrix, C an $(n-k)\times k$ matrix, and $[0]$ the $k\times(n-k)$ zero matrix. Show that the $n\times n$ matrix $\begin{bmatrix} A & C \\ [0] & B \end{bmatrix}$ is invertible if and only if A and B are invertible.

b. Find a formula for the inverse.

2.6 a. Row reduce the matrix A in the margin.

$$\begin{bmatrix} 1 & -1 & 3 & 0 & -2 \\ -2 & 2 & -6 & 0 & 4 \\ 0 & 2 & 5 & -1 & 0 \\ 2 & -6 & -4 & 2 & -4 \end{bmatrix}$$

Matrix A for Exercise 2.6.

b. Let $\vec{\mathbf{v}}_m$, $m = 1, \dots, 5$ be the columns of A. What can you say about the systems of equations

$$[\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \vec{\mathbf{v}}_{k+1} \quad \text{for } k = 1,2,3,4?$$

Exercise 2.6, part b: For example, for $k = 2$ we are asking about the system of equations

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 0 & 2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 5 \\ -4 \end{bmatrix}.$$

2.7 a. For what values of a is the matrix $\begin{bmatrix} 1 & -1 & -1 \\ 0 & a & 1 \\ 2 & a+2 & a+2 \end{bmatrix}$ invertible?

b. For those values, compute the inverse.

2.8 Show that the following two statements are equivalent to saying that a set of vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ is linearly independent:

a. The only way to write the zero vector $\vec{\mathbf{0}}$ as a linear combination of the $\vec{\mathbf{v}}_i$ is to use only zero coefficients.

b. None of the $\vec{\mathbf{v}}_i$ is a linear combination of the others.

2.9 a. Show that $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ form an orthonormal basis of $\mathbb{R}^2$.

b. Show that $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$ form an orthonormal basis of $\mathbb{R}^2$.

c. Show that any orthogonal 2×2 matrix gives either a reflection or a rotation: a reflection if its determinant is negative, a rotation if its determinant is positive.

2.10 Find a bound on $a^2 + b^2$ such that Newton's method to solve

$$x^3 + x - 3y^2 = a$$
$$x^5 + x^2y^3 - y = b$$

starting at $x = 0$, $y = 0$, is sure to converge to a solution.

Hint for Exercise 2.10: Set

$$z_1 = x^2$$
$$z_2 = y^2$$
$$z_3 = z_1^2$$

2.11 a. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Are the elements I, A, A^2, A^3 linearly independent in $\text{Mat}\,(2,2)$? What is the dimension of the subspace $V \subset \text{Mat}\,(2,2)$ that they span? (Recall that $\text{Mat}\,(n,m)$ denotes the set of $n\times m$ matrices.)

b. Show that the set W of matrices $B \in \text{Mat}\,(2,2)$ that satisfy $AB = BA$ is a subspace of $\text{Mat}\,(2,2)$. What is its dimension?

c. Show that $V \subset W$. Are they equal?

2.12 Let $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ be vectors in $\mathbb{R}^n$, and set $V = [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]$.

a. Show that the set $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ is orthogonal if and only if $V^\top V$ is diagonal.

b. Show that the set $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ is orthonormal if and only if $V^\top V = I_k$.

2.13 Find a basis for the image and the kernel of the matrices

$$A = \begin{bmatrix} 1 & 1 & 3 & 6 & 2 \\ 2 & -1 & 0 & 4 & 1 \\ 4 & 1 & 6 & 16 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 3 & 6 & 2 \\ 2 & -1 & 0 & 4 & 1 \end{bmatrix},$$

and verify that the dimension formula (equation 2.5.8) is true.

Exercise 2.14: To lighten notation, we omit arrows on the vectors, writing $\overline{\mathbf{v}}$ for the complex conjugate of $\mathbf{v}$.

2.14 Let A be an $n \times n$ matrix with real entries, and let $\mathbf{v} \in \mathbb{C}^n$ be a vector such that $A\mathbf{v} = (a+ib)\mathbf{v}$ with $b \neq 0$. Let $\mathbf{u} = \frac{1}{2}(\mathbf{v} + \overline{\mathbf{v}})$ and $\mathbf{w} = \frac{1}{2i}(\mathbf{v} - \overline{\mathbf{v}})$.

a. Show that $A\overline{\mathbf{v}} = (a-ib)\overline{\mathbf{v}}$, so that $\mathbf{v}$ and $\overline{\mathbf{v}}$ are linearly independent in $\mathbb{C}^n$.

b. Show that $A\mathbf{u} = a\mathbf{u} - b\mathbf{w}$ and $A\mathbf{w} = b\mathbf{u} + a\mathbf{w}$.

2.15 Show that an orthogonal 2×2 matrix is either a rotation or a reflection.

Exercise 2.16: For example, the polynomial

$$p = 2x - y + 3xy + 5y^2$$

corresponds to the point $\begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 3 \\ 5 \end{pmatrix}$, so

$$xD_1p = x(2+3y) = 2x + 3xy$$

corresponds to $\begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$.

2.16 Let P be the space of polynomials of degree at most 2 in the variables x, y, identifying $a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$ with $\begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix} \in \mathbb{R}^6$.

a. What are the matrices of the linear transformations $S, T : P \to P$

$$S(p)\begin{pmatrix} x \\ y \end{pmatrix} = xD_1p\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad T(p)\begin{pmatrix} x \\ y \end{pmatrix} = yD_2p\begin{pmatrix} x \\ y \end{pmatrix}?$$

b. What are the kernel and the image of the linear transformation

$$p \mapsto 2p - S(p) - T(p)?$$

Hint for Exercise 2.17: You should use the fact that a polynomial p of degree d such that $p(n) = p'(n) = 0$ can be written $p(x) = (x-n)^2q(x)$ for some polynomial q of degree $d-2$.

2.17 Let $a_1, \dots, a_k, b_1, \dots, b_k$ be any $2k$ numbers. Show that there exists a unique polynomial p of degree at most $2k-1$ such that $p(n) = a_n, \quad p'(n) = b_n$ for all integers n with $1 \leq n \leq k$. In other words, show that the values of p and p' at $1, \dots, k$ determine p.

Hint for Exercise 2.18, part b: $\vec{\mathbf{v}} = P\vec{\mathbf{v}} + (\vec{\mathbf{v}} - P\vec{\mathbf{v}})$.

2.18 A square $n \times n$ matrix P such that $P^2 = P$ is called a *projection*.

a. Show that P is a projection if and only if $I - P$ is a projection. Show that if P is invertible, then P is the identity.

b. Let $V_1 = \operatorname{img} P$ and $V_2 = \ker P$. Show that any vector $\vec{\mathbf{v}} \in \mathbb{R}^n$ can be written uniquely $\vec{\mathbf{v}} = \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2$ with $\vec{\mathbf{v}}_1 \in V_1$ and $\vec{\mathbf{v}}_2 \in V_2$.

c. Show that there exist a basis $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n$ of $\mathbb{R}^n$ and a number $k \leq n$ such that

$$P\vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_1,\ P\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_2, \dots, P\vec{\mathbf{v}}_k = \vec{\mathbf{v}}_k \quad \text{and}$$
$$P\vec{\mathbf{v}}_{k+1} = \mathbf{0},\ P\vec{\mathbf{v}}_{k+2} = \vec{\mathbf{0}}, \dots, P\vec{\mathbf{v}}_n = \vec{\mathbf{0}}.$$

*d. Show that if P_1 and P_2 are projections such that $P_1P_2 = [0]$, then $Q = P_1 + P_2 - (P_2P_1)$ is a projection, $\ker Q = \ker P_1 \cap \ker P_2$, and the image of Q is the space spanned by the image of P_1 and the image of P_2.

Exercise 2.19: Recall that $\mathcal{C}^2$ is the space of C^2 (twice continuously differentiable) functions.

2.19 Show that the transformation $T : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ given by formula 2.6.8 in Example 2.6.7 is a linear transformation.

2.20 Denote by $\mathcal{L}\Big(\text{Mat}\,(n,n), \text{Mat}\,(n,n)\Big)$ the space of linear transformations from $\text{Mat}\,(n,n)$ to $\text{Mat}\,(n,n)$.

a. Show that $\mathcal{L}\Big(\text{Mat}\,(n,n), \text{Mat}\,(n,n)\Big)$ is a vector space and that it is finite dimensional. What is its dimension?

b. Prove that for any $A \in \text{Mat}\,(n,n)$, the transformations

$$L_A,\ R_A : \text{Mat}\,(n,n) \to \text{Mat}\,(n,n)$$

given by $L_A(B) = AB$ and $R_A(B) = BA$ are linear transformations.

c. Let $\mathcal{M}_L \subset \mathcal{L}\Big(\text{Mat}\,(n,n), \text{Mat}\,(n,n)\Big)$ be the set of functions of the form L_A. Show that it is a subspace of $\mathcal{L}\Big(\text{Mat}\,(n,n), \text{Mat}\,(n,n)\Big)$. What is its dimension?

d. Show that there are linear transformations $T : \text{Mat}\,(2,2) \to \text{Mat}\,(2,2)$ that cannot be written as $L_A + R_B$ for any two matrices $A, B \in \text{Mat}\,(2,2)$. Can you find an explicit one?

2.21 Show that in a vector space of dimension n, more than n vectors are never linearly independent, and fewer than n vectors never span.

2.22 Suppose we use the same operator $T : P_2 \to P_2$ as in Exercise 2.6.8, but choose instead to work with the basis

$$q_1(x) = x^2, \quad q_2(x) = x^2 + x, \quad q_3(x) = x^2 + x + 1.$$

Now what is the matrix $\Phi^{-1}_{\{q\}} \circ T \circ \Phi_{\{q\}}$?

We thank Tan Lei and Alexandre Bardet for Exercise 2.23.

2.23 Let A be a $k \times n$ matrix. Show that if you row reduce the augmented matrix $[A^\top | I_n]$ to get $[\widetilde{A}^\top \widetilde{B}^\top]$, the nonzero columns of $\widetilde{A}$ form a basis for the image of A, and the nonzero columns of $\widetilde{B}$ form a basis for the kernel of A.

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(x-y) + y^2 \\ \cos(x+y) - x \end{pmatrix}$$

Map for Exercise 2.24

2.24 a. Find a global Lipschitz ratio for the derivative of the map $\mathbf{F}$ defined in the margin.

b. Do one step of Newton's method to solve $\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} .5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, starting at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

c. Can you be sure that Newton's method converges?

Exercise 2.25: Note that

$$[2\,I]^3 = [8\,I], \quad \text{i.e.,}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

2.25 Using Newton's method, solve the equation $A^3 = \begin{bmatrix} 9 & 0 & 1 \\ 0 & 7 & 0 \\ 0 & 2 & 8 \end{bmatrix}$.

Exercise 2.26: The computation really does require you to row reduce a 4×4 matrix.

2.26 Consider the map $F : \text{Mat}\,(2,2) \to \text{Mat}\,(2,2)$ given by $F(A) = A^2 + A^{-1}$. Set $A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B_0 = F(A_0)$, and define

$$U_r = \{\, B \in \text{Mat}\,(2,2) \mid |B - B_0| < r \,\}.$$

Do there exist $r > 0$ and a differentiable mapping $G : U_r \to \text{Mat}\,(2,2)$ such that $F(G(B)) = B$ for every $B \in U_r$?

$$\mathbf{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y - 2 \\ y^2 - x - 6 \end{pmatrix}$$

Map for Exercise 2.27

2.27 a. Find a global Lipschitz ratio for the derivative of the mapping $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given in the margin.

b. Do one step of Newton's method to solve $\mathbf{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ starting at $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

c. Find and sketch a disc in $\mathbb{R}^2$ which you are sure contains a root.

2.28 There are other plausible ways to measure matrices other than the length and the norm; for example, we could declare the size $|A|$ of a matrix A to be the largest absolute value of an entry. In this case, $|A + B| \le |A| + |B|$, but the statement $|A\vec{\mathbf{x}}| \le |A||\vec{\mathbf{x}}|$ (where $|\vec{\mathbf{x}}|$ is the ordinary length of a vector) is false. Find an ϵ so that it is false for

$$A = \begin{bmatrix} 1 & 1 & 1+\epsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Exercise 2.29: The norm $\|A\|$ of a matrix A is defined in Section 2.9 (Definition 2.9.6).

2.29 Show that $\|A\| = \|A^\top\|$.

2.30 In Example 2.10.9 we found that $M = 2\sqrt{2}$ is a global Lipschitz ratio for the function $\mathbf{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(x+y) \\ x^2 - y^2 \end{pmatrix}$. What Lipschitz ratio do you get using the method of second partial derivatives? Using that Lipschitz ratio, what minimum domain do you get for the inverse function at $\mathbf{f}\begin{pmatrix} 0 \\ \pi \end{pmatrix}$?

2.31 a. True or false? The equation $\sin(xyz) = z$ expresses x implicitly as a differentiable function of y and z near the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pi/2 \\ 1 \\ 1 \end{pmatrix}$.

b. True or false? The equation $\sin(xyz) = z$ expresses z implicitly as a differentiable function of x and y near the same point.

Exercise 2.32: You may use the fact that if

$$S : \text{Mat}\,(2,2) \to \text{Mat}\,(2,2)$$

is the squaring map

$$S(A) = A^2,$$

then

$$[\mathbf{D}S(A)]B = AB + BA.$$

2.32 True or false? There exist a neighborhood $U \subset \text{Mat}\,(2,2)$ of $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ and a C^1 mapping $F : U \to \text{Mat}\,(2,2)$ with

$$F\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad \text{and} \quad (F(A))^2 = A.$$

2.33 True or false? There exist $r > 0$ and a differentiable map

$$g : B_r\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right) \to \text{Mat}\,(2,2) \quad \text{such that} \quad g\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$

and $(g(A))^2 = A$ for all $A \in B_r\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right)$.

Dierk Schleicher contributed Exercise 2.34. Geometrically, the condition given is the condition that there exists a unit cube with a vertex at the origin such that the three sides emanating from the origin are $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$.

2.34 Given three vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ in $\mathbb{R}^2$, show that there exist vectors $\vec{\mathbf{a}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \vec{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ in $\mathbb{R}^3$ such that

$$|\vec{\mathbf{a}}|^2 = |\vec{\mathbf{b}}|^2 = |\vec{\mathbf{c}}|^2 = 1 \quad \text{and} \quad \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = 0$$

if and only if $\vec{\mathbf{v}}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ are of unit length and orthogonal.

Exercise 2.35: There are many "right" answers to this question, so try to think of a few.

2.35 Imagine that, when constructing a Newton sequence

$$\mathbf{x}_{n+1} = \mathbf{x}_n - [\mathbf{Df}(\mathbf{x}_n)]^{-1}\mathbf{f}(\mathbf{x}_n),$$

you happen upon a noninvertible matrix $[\mathbf{Df}(\mathbf{x}_n)]$. What should you do? Suggest ways to deal with the situation.

Exercise 2.36: The answer to part d depends on whether you choose the additive identity to be $\{0\}$ or allow it to be whatever is appropriate for the particular subset you are looking at. In the latter case you might land on *quotient spaces*.

2.36 Let V be a vector space, and denote by $\mathcal{P}^*(V)$ the set of nonempty subsets of V. Define $+ : \mathcal{P}^*(V) \times \mathcal{P}^*(V) \to \mathcal{P}^*(V)$ by

$$A + B \stackrel{\text{def}}{=} \{\, a + b \mid a \in A, b \in B \,\}$$

and scalar multiplication $\mathbb{R} \times \mathcal{P}^*(V) \to \mathcal{P}^*(V)$ by $\alpha A \stackrel{\text{def}}{=} \{\, \alpha a \mid a \in A \,\}$.

a. Show that $+$ is associative: $(A + B) + C = A + (B + C)$ and that $\{0\}$ is a an additive identity.

b. Show that $\alpha(A + B) = \alpha A + \alpha B$, $1A = A$, and $(\alpha\beta)A = \alpha(\beta A)$, for all $\alpha, \beta \in \mathbb{R}$.

c. Is $\mathcal{P}^*(V)$ a vector space with these operations?

*d. Does $\mathcal{P}^*(V)$ have subsets that are vector spaces with these operations?

***2.37** This exercise gives a proof of *Bezout's theorem*. Let p_1 and p_2 be polynomials of degree k_1 and k_2 respectively, and consider the mapping

$$T\colon (q_1, q_2) \mapsto p_1 q_1 + p_2 q_2,$$

where q_1 is a polynomial of degree at most $k_2 - 1$ and q_2 is a polynomial of degree at most $k_1 - 1$, so that $p_1 q_1 + p_2 q_2$ is of degree $\leq k_1 + k_2 - 1$. Note that the space of such (q_1, q_2) is of dimension $k_1 + k_2$, and the space of polynomials of degree at most $k_1 + k_2 - 1$ is also of dimension $k_1 + k_2$.

Exercise 2.37, part a: It may be easier to work over the complex numbers.

Relatively prime: with no common factors.

a. Show that $\ker T = \{0\}$ if and only if p_1 and p_2 are *relatively prime*.

b. Use Corollary 2.5.10 to prove *Bezout's identity*: if p_1, p_2 are relatively prime, then there exist unique q_1 and q_2 of degree at most $k_2 - 1$ and $k_1 - 1$ such that $p_1 q_1 + p_2 q_2 = 1$.

****2.38** Let A be an $n \times n$ diagonal matrix: $A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, and suppose that one of the diagonal entries, say λ_k, satisfies

$$\inf_{k \neq j} |\lambda_k - \lambda_j| \geq m > 0$$

for some number m. Let B be an $n \times n$ matrix. Find a number R, depending on m, such that if $|B| < R$, then Newton's method will converge if it is used to solve

$$(A + B)\mathbf{x} = \mu \mathbf{x}, \quad \text{for } \mathbf{x} \text{ satisfying } |\mathbf{x}|^2 = 1,$$

starting at $\mathbf{x}_0 = \vec{\mathbf{e}}_k$, $\mu_0 = \lambda_k$.

Exercise 2.39: For instance, if $n = 4$, then

$$J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Of course if an $n \times n$ matrix A has rank n, then $A = QJ_nP^{-1}$ just says that $A = QP^{-1}$, i.e., A is invertible with inverse PQ^{-1}; see Proposition 1.2.15.

2.39 Prove that any $n \times n$ matrix A of rank k can be written $A = QJ_kP^{-1}$, where P and Q are invertible and $J_k = \begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix}$, where I_k is the $k \times k$ identity matrix.

2.40 Let a sequence of integers $a_0, a_1, a_2, \dots$ be defined inductively by

$$a_0 = 1, \ a_1 = 0, \quad \text{and} \quad a_n = 2a_{n-1} + a_{n-2} \text{ for } n \geq 2.$$

a. Find a matrix M such that $\begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix} = M \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. Express $\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ in terms of powers of M.

b. Find a linear relation between I, M, M^2, and use it to find the eigenvalues of M.

c. Find a matrix P such that $P^{-1}MP$ is diagonal.

d. Compute M^n in terms of powers of numbers. Use the result to find a formula for a_n.

2.41 Exercise 2.2.11 asked you to show that using row reduction to solve n equations in n unknowns takes $n^3 + n^2/2 - n/2$ operations, where a single addition, multiplication, or division counts as one operation. How many operations are needed to compute the inverse of an $n \times n$ matrix A? To perform the matrix multiplication $A^{-1}\vec{\mathbf{b}}$?