0

Preliminaries

Allez en avant, et la foi vous viendra
(Keep going; faith will come.)—Jean d'Alembert (1717–1783), to those questioning calculus

0.0 Introduction

This chapter is intended as a resource. You may be familiar with its contents, or there may be topics you never learned or that you need to review. You should not feel that you need to master Chapter 0 before beginning Chapter 1; just refer back to it as needed. (A possible exception is Section 0.7 on complex numbers.)

We have included reminders in the main text; for example, in Section 1.5 we write, "You may wish to review the discussion of quantifiers in Section 0.2."

In Section 0.1 we share some guidelines that in our experience make reading mathematics easier, and discuss specific issues like sum notation.

Section 0.2 analyzes the rather tricky business of negating mathematical statements. (To a mathematician, the statement "All eleven-legged alligators are orange with blue spots" is an obviously true statement, not an obviously meaningless one.) We first use this material in Section 1.5.

Set theory notation is discussed in Section 0.3. The "eight words" of set theory are used beginning in Section 1.1. The discussion of Russell's paradox is not necessary; we include it because it is fun and not hard.

Section 0.4 defines the word "function" and discusses the relationship between a function being "onto" or "one to one" and the existence and uniqueness of solutions. This material is first needed in Section 1.3.

Most of this text concerns real numbers, but we think that anyone beginning a course in multivariate calculus should know what complex numbers are and be able to compute with them.

Real numbers are discussed in Section 0.5, in particular, least upper bounds, convergence of sequences and series, and the intermediate value theorem. This material is first used in Sections 1.5 and 1.6.

The discussion of countable and uncountable sets in Section 0.6 is fun and not hard. These notions are fundamental to Lebesgue integration.

In our experience, most students studying vector calculus for the first time are comfortable with complex numbers, but a sizable minority have either never heard of complex numbers or have forgotten everything they once knew. If you are among them, we suggest reading at least the first few pages of Section 0.7 and doing some of the exercises.

0.1 Reading mathematics

The most efficient logical order for a subject is usually different from the best psychological order in which to learn it. Much mathematical writing is based too closely on the logical order of deduction in a subject, with too many definitions without, or before, the examples

> *which motivate them, and too many answers before, or without, the questions they address.*—William Thurston

Many students do well in high school mathematics courses without reading their texts. At the college level you are expected to read the book. Better yet, read ahead. If you read a section before listening to a lecture on it, the lecture will be more comprehensible, and if there is something in the text you don't understand, you will be able to listen more actively and ask questions.

Reading mathematics is different from other reading. We think the following guidelines can make it easier. There are two parts to understanding a theorem: understanding the statement, and understanding the proof. *The first is more important than the second.*

What if you don't understand the statement? If there's a symbol in the formula you don't understand, perhaps a δ, look to see whether the next line continues, "where δ is such and such." In other words, read the whole sentence before you decide you can't understand it.

If you're still having trouble, *skip ahead to examples.* This may contradict what you have been told – that mathematics is sequential, and that you must understand each sentence before going on to the next. In reality, although mathematical writing is necessarily sequential, mathematical understanding is not: you (and the experts) never understand perfectly up to some point and not at all beyond. The "beyond", where understanding is only partial, is an essential part of the motivation and the conceptual background of the "here and now". You may often find that when you return to something you left half-understood, it will have become clear in the light of the further things you have studied, even though the further things are themselves obscure.

Many students are uncomfortable in this state of partial understanding, like a beginning rock climber who wants to be in stable equilibrium at all times. To learn effectively one must be willing to leave the cocoon of equilibrium. *If you don't understand something perfectly, go on ahead and then circle back.*

In particular, an example will often be easier to follow than a general statement; you can then go back and reconstitute the meaning of the statement in light of the example. Even if you still have trouble with the general statement, you will be ahead of the game if you understand the examples. We feel so strongly about this that we have sometimes flouted mathematical tradition and given examples before the proper definition.

Read with pencil and paper in hand, making up little examples for yourself as you go on.

Some of the difficulty in reading mathematics is notational. A pianist who has to stop and think whether a given note on the staff is A or F will not be able to sight-read a Bach prelude or Schubert sonata. The temptation, when faced with a long, involved equation, may be to give up. You need to take the time to identify the "notes".

The Greek Alphabet

Greek letters that look like Roman letters are not used as mathematical symbols; for example, A is capital a, not capital α. The letter χ is pronounced "kye" to rhyme with "sky"; φ, ψ, and ξ may rhyme with either "sky" or "tea".

α	A	alpha
β	B	beta
γ	Γ	gamma
δ	Δ	delta
ϵ	E	epsilon
ζ	Z	zeta
η	H	eta
θ	Θ	theta
ι	I	iota
κ	K	kappa
λ	Λ	lambda
μ	M	mu
ν	N	nu
ξ	Ξ	xi
o	O	omicron
π	Π	pi
ρ	P	rho
σ	Σ	sigma
τ	T	tau
υ	Υ	upsilon
φ, ϕ	Φ	phi
χ	X	chi
ψ	Ψ	psi
ω	Ω	omega

Learn the names of Greek letters – not just the obvious ones like alpha, beta, and pi, but the more obscure psi, xi, tau, omega. The authors know a mathematician who calls all Greek letters "xi" (ξ), except for omega (ω), which he calls "w". This leads to confusion. Learn not just to recognize these letters, but how to pronounce them. Even if you are not reading mathematics out loud, it is hard to think about formulas if ξ, ψ, τ, ω, φ are all "squiggles" to you.

Sum and product notation

Sum notation can be confusing at first; we are accustomed to reading in one dimension, from left to right, but something like

$$\sum_{k=1}^{n} a_{i,k} b_{k,j} \qquad 0.1.1$$

requires what we might call two-dimensional (or even three-dimensional) thinking. It may help at first to translate a sum into a linear expression:

$$\sum_{i=0}^{\infty} 2^i = 2^0 + 2^1 + 2^2 \dots \qquad 0.1.2$$

$$\text{or} \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} = a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \cdots + a_{i,n} b_{n,j}. \qquad 0.1.3$$

In equation 0.1.3, the symbol $\sum_{k=1}^{n}$ says that the sum will have n terms. Since the expression being summed is $a_{i,k} b_{k,j}$, each of those n terms will have the form ab.

Usually the quantity being summed has an index matching the index of the sum (for instance, k in formula 0.1.1). If not, it is understood that you add one term for every "whatever" that you are summing over. For example, $\sum_{i}^{10} 1 = 10$.

Two $\sum$ placed side by side do not denote the product of two sums; one sum is used to talk about one index, the other about another. The same thing could be written with one $\sum$, with information about both indices underneath. For example,

$$\begin{aligned} \sum_{i=1}^{3} \sum_{j=2}^{4} (i+j) &= \sum_{\substack{i \text{ from } 1 \text{ to } 3, \\ j \text{ from } 2 \text{ to } 4}} (i+j) \\ &= \left(\sum_{j=2}^{4} 1+j \right) + \left(\sum_{j=2}^{4} 2+j \right) + \left(\sum_{j=2}^{4} 3+j \right) \qquad 0.1.4 \\ &= \big((1+2)+(1+3)+(1+4) \big) \\ &\quad + \big((2+2)+(2+3)+(2+4) \big) \\ &\quad + \big((3+2)+(3+3)+(3+4) \big); \end{aligned}$$

this double sum is illustrated in Figure 0.1.1.

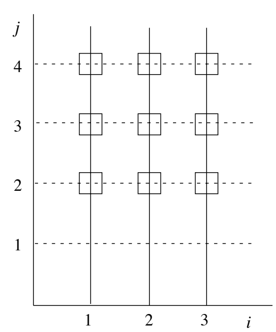

FIGURE 0.1.1.

In the double sum of equation 0.1.4, each sum has three terms, so the double sum has nine terms.

Rules for product notation $\prod$ are analogous to those for sum notation:

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n; \quad \text{for example,} \quad \prod_{i=1}^{n} i = n!. \qquad 0.1.5$$

Proofs

When Jacobi complained that Gauss's proofs appeared unmotivated, Gauss is said to have answered, *You build the building and remove the scaffolding.* Our sympathy is with Jacobi's reply: he likened Gauss to *the fox who erases his tracks in the sand with his tail.*

FIGURE 0.1.2.
Nathaniel Bowditch (1773–1838)

We said earlier that it is more important to understand a mathematical statement than to understand its proof. We have put some of the harder proofs in Appendix A; these can safely be skipped by a student studying multivariate calculus for the first time. We urge you, however, to read the proofs in the main text. By reading many proofs you will learn what a proof is, so that (for one thing) you will know when you have proved something and when you have not.

In addition, a good proof doesn't just convince you that something is true; it tells you why it is true. You presumably don't lie awake at night worrying about the truth of the statements in this or any other math textbook. (This is known as "proof by eminent authority": you assume the authors know what they are talking about.) But reading the proofs will help you understand the material.

If you get discouraged, keep in mind that the contents of this book represent a cleaned-up version of many false starts. For example, John Hubbard started by trying to prove Fubini's theorem in the form presented in equation 4.5.1. When he failed, he realized (something he had known and forgotten) that the statement was in fact false. He then went through a stack of scrap paper before coming up with a correct proof. Other statements in the book represent the efforts of some of the world's best mathematicians over many years.

0.2 Quantifiers and negation

According to a contemporary, the French mathematician Laplace (1749–1827) wrote *il est aisé à voir* ("it's easy to see") whenever he couldn't remember the details of a proof.

"I never come across one of Laplace's '*Thus it plainly appears*' without feeling sure that I have hours of hard work before me to fill up the chasm and find out and show *how* it plainly appears," wrote Bowditch.

Forced to leave school at age 10 to help support his family, the American Bowditch taught himself Latin in order to read Newton, and French in order to read French mathematics. He made use of a scientific library captured by a privateer and taken to Salem. In 1806 he was offered a professorship at Harvard but turned it down.

Interesting mathematical statements are seldom like "$2 + 2 = 4$"; more typical is the statement "every prime number such that if you divide it by 4 you have a remainder of 1 is the sum of two squares." In other words, most interesting mathematical statements are about infinitely many cases; in the case above, it is about all those prime numbers such that if you divide them by 4 you have a remainder of 1 (there are infinitely many such numbers).

In a mathematical statement, every variable has a corresponding quantifier, either implicit or explicitly stated. There are two such quantifiers: "for all" (the *universal quantifier*), written symbolically $\forall$, and "there exists" (the *existential quantifier*), written $\exists$. Above we have a single quantifier, "every". More complicated statements have several quantifiers, for example, the statement, "For all $x \in \mathbb{R}$ and for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$." This true statement says that the squaring function is continuous.

The order in which these quantifiers appears matters. If we change the order of quantifiers in the preceding statement about the squaring function to "For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$," we have a meaningful mathematical sentence but it is false. (It claims that the squaring function is uniformly continuous, which it is not.)