Preface

Joseph Fourier (1768–1830)

Fourier was arrested during the French Revolution and threatened with the guillotine, but survived and later accompanied Napoleon to Egypt; in his day he was as well known for his studies of Egypt as for his contributions to mathematics and physics. He found a way to solve linear partial differential equations while studying heat diffusion. An emphasis on computationally effective algorithms is one theme of this book.

... The numerical interpretation ... is however necessary. ... So long as it is not obtained, the solutions may be said to remain incomplete and useless, and the truth which it is proposed to discover is no less hidden in the formulae of analysis than it was in the physical problem itself.

–Joseph Fourier, *The Analytic Theory of Heat*

Chapters 1 through 6 of this book cover the standard topics in multivariate calculus and a first course in linear algebra. The book can also be used for a course in analysis, using the proofs in the appendix.

The organization and selection of material differs from the standard approach in three ways, reflecting the following principles.

> *First, we believe that at this level linear algebra should be more a convenient setting and language for multivariate calculus than a subject in its own right. The guiding principle of this unified approach is that locally, a nonlinear function behaves like its derivative.*

Thus when we have a question about a nonlinear function we answer it by looking carefully at a linear transformation: its derivative. In this approach, everything learned about linear algebra pays off twice: first for understanding linear equations, then as a tool for understanding nonlinear equations.

We discuss abstract vector spaces in section 2.6, but the emphasis is on $\mathbb{R}^n$, as we believe that most students find it easiest to move from the concrete to the abstract.

> *Second, we emphasize computationally effective algorithms, and we prove theorems by showing that these algorithms work.*

We feel this better reflects the way mathematics is used today, in both applied and pure mathematics. Moreover, it can be done with no loss of rigor.

For linear equations, row reduction is the central tool from which everything else follows; we use row reduction to prove all the standard results about dimension and rank. For nonlinear equations, the cornerstone is Newton's method, the best and most widely used method for solving nonlinear equations; we use it both as a computational tool and in proving the inverse and implicit function theorems. We include a section on numerical methods of integration, and we encourage the use of computers both to reduce tedious calculations and as an aid in visualizing curves and surfaces.

> *Third, we use differential forms to generalize the fundamental theorem of calculus to higher dimensions.*

The great conceptual simplification gained by doing electromagnetism in the language of forms is a central motivation for using forms. We apply the language of forms to electromagnetism and potentials in sections 6.11 and 6.12, which are expanded in this fourth edition.

In our experience, differential forms can be taught to freshmen and sophomores if forms are presented geometrically, as integrands that take an oriented piece of a curve, surface, or manifold, and return a number. We are aware that students taking courses in other fields need to master the language of vector calculus, and we devote three sections of chapter 6 to integrating the standard vector calculus into the language of forms.

Other ways this book differs from standard texts include

The treatment of eigenvectors and eigenvalues
Rules for computing Taylor polynomials
Lebesgue integration

Eigenvectors and eigenvalues In keeping with our prejudice in favor of computationally effective algorithms, we provide in section 2.7 a theory of eigenvectors and eigenvalues that bypasses determinants, which are more or less uncomputable for large matrices. The treatment we give is also stronger theoretically: theorem 2.7.6 gives an "if and only if" statement for the existence of eigenbases. In addition, our emphasis on defining an eigenvector $\mathbf{v}$ as satisfying $A\mathbf{v} = \lambda\mathbf{v}$ has the advantage of working when A is a linear transformation between infinite-dimensional vector spaces, whereas the definition in terms of roots of the characteristic polynomial does not. However, in section 4.8 we define the characteristic polynomial of a matrix, connecting eigenvalues and eigenvectors to determinants.

Rules for computing Taylor polynomials Even good graduate students are often unaware of the rules that make computing Taylor polynomials in higher dimensions palatable. We give these in section 3.4, with proofs in appendix A.11.

In our experience, undergraduates, even freshmen, are quite prepared to approach the Lebesgue integral via the Riemann integral, but the approach via measurable sets and σ-algebras is inconceivable.

Lebesgue integration We give a new approach to Lebesgue integration, tying it much more closely to Riemann integrals, and completely avoiding the standard σ-algebras of measurable sets. We were motivated by two considerations. First, integrals over unbounded domains and integrals of unbounded functions are really important, for instance in physics and probability, and students will need to know about such integrals before they take a course in analysis. Second, there simply does not appear to be a successful theory of improper multiple integrals.

What's new in the fourth edition

The main impetus for producing a new edition rather than reprinting the third edition was that we finally hit on what we consider the right way to define orientation of manifolds. The new approach, based on direct bases, is simpler than the previous, but still covers the case of 0-dimensional manifolds (i.e., points).

The new approach to orientation took less space, which opened up the possibility of adding a proof of Gauss's *remarkable theorem*. This theorem, also known as the "Theorema Egregium", justifies the statement (see section 3.8) that Gaussian curvature measures to what extent pieces of a surface can be made flat, without stretching or deformation. All the other proofs we know of this theorem require advanced techniques: either the Levi-Civita connection associated to the Riemann metric on S inherited from the embedding, or the Jacobi second variation equation. We were pleased with the proof we eventually added to section 5.4. It uses only the techniques developed in this book.

The proof, however, took up more space than we had gained with orientation. In the third edition, we kept to the 802-page count of the second edition. For this edition we allowed ourselves a few extra pages. This enabled us to include all of the material on our wish list, including

1. a justification of the statement in section 3.8 that the mean curvature measures how far a surface is from being minimal
2. a discussion of Faraday's experiments in section 6.11 on electromagnetism
3. a trick for finding Lipschitz ratios for polynomial functions, adding variables to make the function quadratic (see example 2.8.12)
4. classifying constrained critical points using the *augmented Hessian matrix* (section 3.7)
5. a proof of Poincaré's lemma for arbitrary forms rather than just 1-forms, based on the *cone operator* (section 6.12).

A number of new figures and exercises have also been added; the labels in the figures have been standardized, to make them more attractive and readable. Of course we have corrected all errors on the errata list for the third edition.

A student solution manual, with solutions to odd-numbered exercises, is available from Matrix Editions.

Practical information

Chapter 0 and back cover Chapter 0 is intended as a resource. Students should not feel that they need to read it before beginning chapter 1. Another resource is the inside back cover, which lists some useful formulas.

Errata Errata will be posted at

http://www.MatrixEditions.com

Exercises Exercises are given at the end of each section; chapter review exercises are given at the end of each chapter, except chapter 0 and the appendix. They range from very easy exercises intended to make students familiar with vocabulary, to quite difficult ones. The hardest exercises are marked with an asterisk (in rare cases, two asterisks).

Notation Mathematical notation is not always uniform. For example, $|A|$ can mean the length of a matrix A (the usage in this book) or the determinant of A (not in this book). Different notations for partial derivatives also exist. This should not pose a problem for readers who begin at the beginning and end at the end, but for those who are using only selected chapters, it could be confusing. Notations used in the book are listed on the front inside cover, along with an indication of where they are first introduced.

We often refer back to theorems, examples, and so on, and believe this numbering makes them easier to find.

Numbering Theorems, lemmas, propositions, corollaries, and examples share the same numbering system: proposition 2.3.7 is not the seventh proposition of section 2.3; it is the seventh numbered item of that section.

Figures and tables share their own numbering system; figure 4.5.2 is the second figure or table of section 4.5. Virtually all displayed equations are numbered, with the numbers given at right; equation 4.2.3 is the third equation of section 4.2. When an equation is displayed a second time, it keeps its original number, but the number is in parentheses.

The SAT test used to have a section of analogies; the "right" answer sometimes seemed contestable. In that spirit,

Calculus is to analysis as a sonata is to a Chopin etude (playing a sonata is more fun, the etude will do more to improve your technique).

Calculus is to analysis as playing a sonata is to composing one (composing requires a better understanding of music theory).

Calculus is to analysis as performing in a ballet or modern dance is to choreographing it.

These are meant to suggest that generally, calculus is more fun, and it is challenging – rote learning isn't enough. Analysis involves more painstaking technical work, which at times may seem like drudgery, but it provides a level of mastery that calculus alone cannot give.

Programs The three programs used in this book – NEWTON.M (used in section 2.8), MONTE CARLO (section 4.6), and DETERMINANT (section 4.8) – are posted at http://MatrixEditions.com/Programs.html. NEWTON.M is a MATLAB program; the others are written in Pascal. Readers are welcome to propose additional programs (or translations of the above programs into other programming languages); if interested, please write jhh8@cornell.edu.

Symbols We use $\triangle$ to mark the end of an example or remark, and $\square$ to mark the end of a proof. Sometimes we specify what proof is being ended: for instance, $\square$ corollary 1.6.15 means end of the proof of corollary 1.6.15.

Using this book as a calculus text or as an analysis text

This book can be used at different levels of rigor. Chapters 1 through 6 contain material appropriate for a course in linear algebra and multivariate calculus. Appendix A contains the technical, rigorous underpinnings appropriate for a course in analysis. It includes proofs of those statements not proved in the main text, and a painstaking justification of arithmetic.

In deciding what to include in this appendix, and what to put in the main text, we used the analogy that learning calculus is like learning to drive a car with standard transmission – acquiring the understanding and intuition to shift gears smoothly when negotiating hills, curves, and the stops and starts of city streets. Analysis is like designing and building a car. To use this book to "learn how to drive," appendix A should be omitted.

Most of the proofs included in this appendix are more difficult than the proofs in the main text, but difficulty was not the only criterion; many students find the proof of the fundamental theorem of algebra (section 1.6) quite difficult. But we find this proof qualitatively different from the proof of the Kantorovich theorem, for example. A professional mathematician who has understood the proof of the fundamental theorem of algebra should be able to reproduce it. A professional mathematician who has read through the proof of the Kantorovich theorem, and who agrees that each step is justified, might well want to refer to notes in order to reproduce it. In this sense, the first proof is more conceptual, the second more technical.

Jean Dieudonné (1906–1992)

Dieudonné, one of the founding members of "Bourbaki," a group of young mathematicians who published collectively under the pseudonym Nicolas Bourbaki, and whose goal was to put modern mathematics on a solid footing, was the personification of rigor in mathematics. Yet in his book *Infinitesimal Calculus*, he put the harder proofs in small type, saying "a beginner will do well to accept plausible results without taxing his mind with subtle proofs."

One-year courses

At Cornell University this book is used for the honors courses Math 2230 (fall semester) and 2240 (spring semester). Students are expected to have a 5 on the Advanced Placement BC Calculus exam, or the equivalent. When John Hubbard teaches the course, he typically gets to the middle of chapter 4 in the first semester, sometimes skipping section 3.8 on the geometry of curves and surfaces, and going through sections 4.2–4.4 rather rapidly, in order to get to section 4.5 on Fubini's theorem and begin to compute integrals. In the second semester he gets to the end of chapter 6 and goes on to teach some of the material that will appear in a sequel volume, in particular differential equations.[1]

One could also spend a year on chapters 1–6. Some students might need to review chapter 0; others may be able to include some proofs from the appendix.

Semester courses

1. A one-semester course for students who have studied neither linear algebra nor multivariate calculus.

For such a course, we suggest covering only the first four chapters. Topics that could be omitted include the proof of the fundamental theorem of algebra in section 1.6, criteria for differentiability in section 1.9, the part of section 2.9 concerning a stronger version of the Kantorovich theorem, section 3.8 on the geometry of curves and surfaces, section 4.4 on measure 0 (if Lebesgue integration is to be skipped), the proof of theorem 4.9.1, and the discussion in section 4.11 on Fourier and Laplace transforms.

Sections 4.2 (integrals and probability) and 4.6 (numerical methods of integration) could also be skipped, but we feel these topics are generally given too little attention. If section 4.2 is skipped, then one should also skip the discussion of Monte Carlo methods in section 4.6.

2. A course for students who have had some exposure to either linear algebra or multivariate calculus, but who are not ready for a course in analysis.

[1]Eventually, he would like to take three semesters to cover chapters 1–6 of the current book and the material of the sequel (referred to as "Volume 2" in this text), including differential equations, inner products (with Fourier analysis and wavelets), and advanced topics in differential forms.

We used an earlier version of this text with students who had taken a course in linear algebra, and feel they gained a great deal from seeing how linear algebra and multivariate calculus mesh. Such students could be expected to cover chapters 1–6, possibly omitting some material. For a less fast-paced course, the book could also be covered in a year, possibly including some proofs from the appendix.

3. A one-semester analysis course.

In one semester one could hope to cover all six chapters and some or most of the proofs in the appendix. This could be done at varying levels of difficulty; students might be expected to follow the proofs, for example, or they might be expected to understand them well enough to construct similar proofs.

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Barbara Burke Hubbard (BA Harvard University) is the author of *The World According to Wavelets*, which was awarded the prix d'Alembert by the French Mathematical Society in 1996.