6

Forms and vector calculus

Gradient a 1-form? How so? Hasn't one always known the gradient as a vector? Yes, indeed, but only because one was not familiar with the more appropriate 1-form concept.—C. Misner, K. S. Thorne, J. Wheeler, *Gravitation*

6.0 Introduction

What really makes calculus work is the fundamental theorem of calculus: that differentiation, having to do with speeds, and integration, having to do with areas, are somehow inverse operations.

We want to generalize the fundamental theorem of calculus to higher dimensions. Unfortunately, we cannot do so with the techniques of chapters 4 and 5, where we integrated using $|d^n\mathbf{x}|$. The reason is that $|d^n\mathbf{x}|$ always returns a positive number; it does not concern itself with the orientation of the subset over which it is integrating, unlike the dx of one-dimensional calculus, which does:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \qquad 6.0.1$$

In one-variable calculus, the standard integrand $f(x)\,dx$ takes a piece $[x_i, x_{i+1}]$ of the domain and returns the number

$$f(x_i)(x_{i+1} - x_i):$$

the area of a rectangle with height $f(x_i)$ and width $x_{i+1} - x_i$. Note that dx returns $x_{i+1} - x_i$, not $|x_{i+1} - x_i|$; this accounts for equation 6.0.1.

In chapter 4 we studied the integrand $|d^n\mathbf{x}|$, which takes a (flat) subset $A \subset \mathbb{R}^n$ and returns its n-dimensional volume. In chapter 5 we showed how to integrate $|d^k\mathbf{x}|$ over a (curvy) k-dimensional manifold in $\mathbb{R}^n$ to determine its k-dimensional volume. Such integrands require no mention of the orientation of the piece.

The cancellations due to opposite orientations make possible the fundamental theorem of calculus. To get a fundamental theorem of calculus in higher dimensions, we need to define orientation in higher dimensions, and we need an integrand that gives one number when integrating over a domain with one orientation, and the opposite number when integrating over a domain with the opposite orientation.

It follows that orientation in higher dimensions must be defined in such a way that choosing an orientation is always a choice between one orientation and its opposite. It is fairly clear that you can orient a curve by drawing an arrow on it; orientation then means, what direction are you going along the curve, with the arrow or against it? For a surface in $\mathbb{R}^3$, an orientation is a specification of a direction in which to go through the surface, such as crossing a sphere "from the inside to the outside" or "from the outside to the inside." These two notions of orientation, for a curve and for a surface, but are actually two instances of a single notion: we will provide a single definition of orientation that covers these cases and all others as well (including 0-manifolds, or points, which in other approaches to orientation are sometimes left out).

Differential forms are a special case of *tensors*. A tensor on a manifold is "anything you can build out of tangent vectors and duals of tangent vectors": a vector field is a tensor, as is a quadratic form on tangent vectors. Although tensor calculus is a powerful tool, especially in computations, we find that speaking of tensors tends to hide the nature of the objects under consideration.

Once we have determined how to orient our objects, we must choose our *integrands*: the mathematical creature that assigns a little number to a little piece of the domain. If we were willing to restrict ourselves to $\mathbb{R}^2$ and $\mathbb{R}^3$, we could use the techniques of vector calculus. Instead we will again use forms (also known as *differential forms*). Because forms work in any dimension, they are the natural way to approach two towering subjects that are inherently four-dimensional: electromagnetism and the theory of relativity. Electromagnetism is the subject of section 6.11; section 6.12 introduces the *cone operator* to deal with potentials in sufficient generality to apply to electromagnetism.

Section 6.11 is an ambitious treatment of electromagnetism using forms; we will see that Maxwell's laws can be written in the elegant form

$$\mathbf{d}\mathbb{F} = 0, \quad \mathbf{d}\mathbb{M} = 4\pi\mathbb{J}.$$

Our treatment of forms, especially the exterior derivative, was influenced by Vladimir Arnold's book *Mathematical Methods of Classical Mechanics*.

Forms also make possible a unified treatment of differentiation and of the fundamental theorem of calculus: one operator (the *exterior derivative*) works in all dimensions, and one short, elegant statement (the generalized Stokes's theorem) generalizes the fundamental theorem of calculus to all dimensions. In contrast, vector calculus requires special formulas, operators, and theorems for each dimension where it works.

On the other hand, the language of vector calculus is used in many science courses, particularly at the undergraduate level. If you are studying physics, you definitely need to know vector calculus. In addition, the functions and vector fields of vector calculus are more intuitive than forms. A vector field is an object that one can picture, as in figure 6.0.1. Coming to terms with forms requires more effort. We can't draw you a picture of a form. A k-form is, as we shall see, something like the determinant: it takes k vectors, fiddles with them until it has a square matrix, and then takes its determinant.

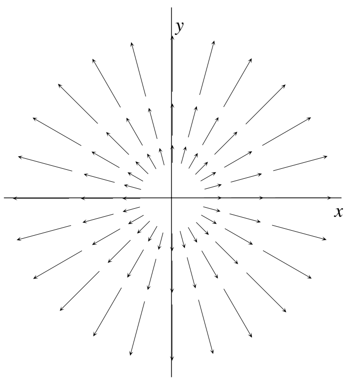

FIGURE 6.0.1.
The radial vector field
$$\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

For these two reasons we have devoted three sections to translating between forms and vector calculus: section 6.5 relates forms on $\mathbb{R}^3$ to functions and vector fields, section 6.8 shows that the exterior derivative we define using forms has three separate incarnations in the language of vector calculus, and section 6.10 shows how Stokes's theorem, a single statement in the language of forms, becomes four more complicated statements in the language of vector calculus.

We begin by introducing forms; we will then see (section 6.2) how to integrate forms over parametrized domains (domains that come with an inherent orientation), before tackling the issue of orientation in sections 6.3 and 6.4.

6.1 Forms on $\mathbb{R}^n$

In section 4.8 we saw that the determinant is the unique antisymmetric and multilinear function of n vectors in $\mathbb{R}^n$ that gives 1 if evaluated on the standard basis vectors. Because of the connection between the determinant and volume described in section 4.9, the determinant is fundamental to changes of variables in multiple integrals, as we saw in section 4.10.

The important difference between determinants and k-forms is that a k-form on $\mathbb{R}^n$ is a function of k vectors, while the determinant on $\mathbb{R}^n$ is a function of n vectors; determinants are defined only for square matrices.

Here we will study the multilinear antisymmetric functions of k vectors in $\mathbb{R}^n$, where $k \geq 0$ may be any integer, though we will see that the only interesting case is when $k \leq n$. Again there is a close relation to volumes; these objects, called *forms*, are the right integrands for integrating over oriented domains.

Definition 6.1.1 (k-form on $\mathbb{R}^n$). A *k-form* on $\mathbb{R}^n$ is a function φ that takes k vectors in $\mathbb{R}^n$ and returns a number $\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$, such that φ is multilinear and antisymmetric as a function of the vectors.

"Antisymmetric" and "alternating" are synonymous.

Antisymmetry

If you exchange any two of the arguments of φ, you change the sign of φ:

$$\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_i, \dots, \vec{\mathbf{v}}_j, \dots, \vec{\mathbf{v}}_k) = -\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_j, \dots, \vec{\mathbf{v}}_i, \dots, \vec{\mathbf{v}}_k).$$

Multilinearity

If φ is a k-form and

$$\vec{\mathbf{v}}_i = a\vec{\mathbf{u}} + b\vec{\mathbf{w}},$$

then

$$\varphi\Big(\vec{\mathbf{v}}_1, \dots, (a\vec{\mathbf{u}} + b\vec{\mathbf{w}}), \dots, \vec{\mathbf{v}}_k\Big) = a\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{i-1}, \vec{\mathbf{u}}, \vec{\mathbf{v}}_{i+1}, \dots, \vec{\mathbf{v}}_k) + b\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{i-1}, \vec{\mathbf{w}}, \vec{\mathbf{v}}_{i+1}, \dots, \vec{\mathbf{v}}_k).$$

The number k is called the *degree* of the form.

The next example is the fundamental example.

Example 6.1.2 (k-form). Let $i_1, \dots, i_k$ be any k integers between 1 and n. Then $dx_{i_1} \wedge \dots \wedge dx_{i_k}$ is that function of k vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ in $\mathbb{R}^n$ that puts these vectors side by side, making the $n \times k$ matrix

$$\begin{bmatrix} v_{1,1} & \dots & v_{1,k} \\ \vdots & \dots & \vdots \\ v_{n,1} & \dots & v_{n,k} \end{bmatrix} \qquad 6.1.1$$

and selects k rows: first row i_1, then row i_2, etc., and finally row i_k, making a square $k \times k$ matrix, and finally takes its determinant. For instance,

$$\underbrace{dx_1 \wedge dx_2}_{\text{2-form}} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 2 \end{bmatrix} \right) = \det \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}}_{\substack{\text{1st and 2nd rows} \\ \text{of original matrix}}} = -8. \qquad 6.1.2$$

$$\underbrace{dx_1 \wedge dx_2 \wedge dx_4}_{\text{3-form}} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right) = \det \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 1 \\ 1 & 2 & 1 \end{bmatrix} = -7 \qquad 6.1.3$$

$$\underbrace{dx_2 \wedge dx_1 \wedge dx_4}_{\text{3-form}} \left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right) = \det \begin{bmatrix} 2 & -2 & 1 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = 7 \qquad \triangle$$

Equation 6.1.3: Note that to give an example of a 3-form we had to add a third vector. You cannot evaluate a 3-form on two vectors (or on four); a k-form is a function of k vectors. But you *can* evaluate a 2-form on two vectors in $\mathbb{R}^4$ (as we did in equation 6.1.2) or in $\mathbb{R}^{16}$. This is not the case for the determinant, which is a function of n vectors in $\mathbb{R}^n$.

Example 6.1.3 (0-form). Definition 6.1.1 makes sense even if $k = 0$: a 0-form on $\mathbb{R}^n$ takes no vectors and returns a number. In other word, it is that number. $\triangle$

Remarks. 1. For now think of a form like $dx_1 \wedge dx_2$ or $dx_1 \wedge dx_2 \wedge dx_4$ as a single item, without worrying about the component parts. The reason for the wedge $\wedge$ will be explained at the end of this section, where we discuss the *wedge product*; we will see that the use of $\wedge$ in the wedge product is consistent with its use here. In section 6.8 we will see that the use of d in our notation here is consistent with its use to denote the *exterior derivative.*

2. The integrand $|d^k\mathbf{x}|$ of chapter 5 also takes k vectors in $\mathbb{R}^n$ and gives a number:

$$|d^k\mathbf{x}|(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \sqrt{\det\Big([\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]^\top [\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k]\Big)}. \qquad 6.1.4$$

But these integrands are neither multilinear nor antisymmetric. $\triangle$

Note there are no nonzero k-forms on $\mathbb{R}^n$ when $k > n$. If $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ are vectors in $\mathbb{R}^n$ and $k > n$, then the vectors are not linearly independent, and at least one of them is a linear combination of the others, say

$$\vec{\mathbf{v}}_k = \sum_{i=1}^{k-1} a_i \vec{\mathbf{v}}_i. \qquad 6.1.5$$

Then if φ is a k-form on $\mathbb{R}^n$, evaluation on $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ gives

$$\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \varphi(\vec{\mathbf{v}}_1, \dots, \sum_{i=1}^{k-1} a_i \vec{\mathbf{v}}_i) = \sum_{i=1}^{k-1} a_i \varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{k-1}, \vec{\mathbf{v}}_i). \qquad 6.1.6$$

The first term of the sum at right is $a_1\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{k-1}, \vec{\mathbf{v}}_1)$, the second is $a_2\varphi(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_{k-1}, \vec{\mathbf{v}}_2)$, and so on; each term evaluates φ on k vectors, two of which are equal, and so (by antisymmetry) the k-form returns 0.

Geometric meaning of k-forms

Evaluating the 2-form $dx_1 \wedge dx_2$ on the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, we have

$$dx_1 \wedge dx_2 \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1. \qquad 6.1.7$$

Rather than imagining projecting $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ onto the plane to get the vectors of equation 6.1.8, we could imagine projecting the parallelogram spanned by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ onto the plane to get the parallelogram spanned by the vectors of equation 6.1.8.

If we project $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ onto the (x_1, x_2)-plane, we get the vectors

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}; \qquad 6.1.8$$

the determinant in equation 6.1.7 gives the *signed area of the parallelogram that they span.*

> *Thus $dx_1 \wedge dx_2$ deserves to be called the (x_1, x_2)-component of signed area. Similarly, $dx_2 \wedge dx_3$ and $dx_1 \wedge dx_3$ deserve to be called the (x_2, x_3)- and (x_1, x_3)-components of signed area.*

We can now interpret equations 6.1.2 and 6.1.3 geometrically. The 2-form $dx_1 \wedge dx_2$ tells us that the (x_1, x_2)-component of signed area of the parallelogram spanned by the two vectors in equation 6.1.2 is -8. The 3-form $dx_1 \wedge dx_2 \wedge dx_4$ tells us that the (x_1, x_2, x_4)-component of signed volume of the parallelepiped spanned by the three vectors in equation 6.1.3 is -7.

Similarly, the 1-form dx gives the x-component of signed length of a vector, while dy gives its y-component:

$$dx \left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right) = \det 2 = 2 \quad \text{and} \quad dy \left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right) = \det(-3) = -3. \qquad 6.1.9$$

More generally (and an advantage of k-forms is that they generalize so easily to higher dimensions), we see that

$$dx_i \left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right) = \det[v_i] = v_i \qquad 6.1.10$$