b. Is the vector field $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ on $\mathbb{R}^3$ the curl of another vector field?

6.12.5 a. Compute $\mathbf{c}(\varphi)$ if $\varphi = x\,dx + y\,dy$, and check that $\mathbf{dc}\varphi = \varphi$.

b. Compute $\mathbf{c}(\varphi)$ if $\varphi = x\,dy - y\,dx$, and check that $\mathbf{dc}\varphi + \mathbf{cd}\varphi = \varphi$.

6.12.6 Use the properties of multilinearity and antisymmetry to justify equation 6.12.21.

6.12.7 a. Find the Faraday 2-form $\mathbb{F}$ of the electromagnetic field defined by the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ of exercise 6.11.8.

b. Using the cone operator, find a potential $\mathbb{A}$ for this $\mathbb{F}$.

c. Check that $\mathbf{d}\mathbb{A} = \mathbb{F}$.

****6.12.8** a. Show that a 1-form φ on $\mathbb{R}^2 - \{\mathbf{0}\}$ can be written $\mathbf{d}f$ exactly when $\mathbf{d}\varphi = 0$ and $\int_{S^1} \varphi = 0$, where S^1 is the unit circle, oriented counterclockwise.

b. Show that a 1-form φ on $\mathbb{R}^2 - \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ can be written $\mathbf{d}f$ exactly when $\mathbf{d}\varphi = 0$ and both $\int_{S_1} \varphi = 0$, $\int_{S_2} \varphi = 0$, where S_1 is the circle of radius 1/2 centered at the origin and S_2 is the circle of radius 1/2 centered at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, both oriented counterclockwise.

Exercise 6.12.9, part a: By "proper detail and notation" we mean replace (lim) with the correct description of each limit and replace P_h P by the correct notation for the appropriate parallelograms.

6.12.9 a. Rewrite equation 6.12.30 with the proper detail and notation.

b. Carefully justify each equality in equation 6.12.30.

6.12.10 Confirm the computations for g and h in equation 6.12.27.

6.13 Review exercises for chapter 6

In exercise 6.1, B is an $n \times n$ matrix and φ and ψ are both 1-forms on $\mathbb{R}^3$; $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are vectors in $\mathbb{R}^3$; f is integrable.

6.1 Which of the following are numbers? Identify those that are not.

a.	$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$	b.	$dx_1 \wedge dx_2(\vec{\mathbf{v}}, \vec{\mathbf{w}})$	c.	$\vec{\mathbf{v}} \times \vec{\mathbf{w}}$	d.	$\det B$
e.	$\operatorname{rank} B$	f.	$\operatorname{tr} B$	g.	$\dim A^k(\mathbb{R}^n)$	h.	$\lvert\vec{\mathbf{v}}\rvert$
i.	$A^k(\mathbb{R}^k)$	j.	$\varphi \wedge \psi(\vec{\mathbf{v}}, \vec{\mathbf{w}})$	k.	$\int_{\mathbb{R}} f(x)\,dx$	l.	$\operatorname{sgn}(\sigma)$

6.2 Let $\vec{F}$ be a vector field in $\mathbb{R}^n$, let $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}$ be vectors in $\mathbb{R}^n$, and let φ be the $(n-1)$-form on $\mathbb{R}^n$ given by

$$\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}) = \det[\vec{F}(\mathbf{x}), \vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{n-1}].$$

Write φ as a linear combination of elementary $(n-1)$-forms on $\mathbb{R}^n$, in terms of the coordinates of $\vec{F}$.

6.3 Use definition 6.1.12 to write the wedge product $\varphi \wedge \psi(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4)$, where φ is a 1-form and ψ is a 3-form, as a combination of values of φ and ψ evaluated on appropriate vectors (as in equations 6.1.30 and 6.1.35).

6.4 Set up each of the following integrals of form fields over parametrized domains as an ordinary multiple integral.

a. $\int_{[\gamma(I)]} y^2 dy + x^2 dz$, where $I = [0,a]$ and $\gamma(t) = \begin{pmatrix} t^3 \\ t^2+1 \\ t^2-1 \end{pmatrix}$

b. $\int_{[\gamma(U)]} \sin y^2\, dx \wedge dz$, where $U = [0,a] \times [0,b]$, and $\gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^2 - v \\ uv \\ v^4 \end{pmatrix}$

6.5 Find a 1-form field on $\mathbb{R}^2$ whose sign orients the circle of radius 1 centered at $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ in the clockwise direction.

6.6 Find a 2-form whose sign orients the surface $S \subset \mathbb{R}^3$ given by

$$x^2 + y^3 + z^4 = 1.$$

6.7 Consider the manifold $S^3 \subset \mathbb{R}^4$ of equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

a. Show that $\operatorname{sgn} dx_1 \wedge dx_2 \wedge dx_3$ is not an orientation of S^3.

b. Show that $\Omega_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3) = \operatorname{sgn} \det[\mathbf{x}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$ is an orientation.

6.8 a. Show that the locus $M \subset \mathbb{R}^4$ given by $x_1^2 + x_2^2 + x_1 x_4^2 = 1$ is a smooth manifold.

b. Find a 3-form field whose sign orients M.

6.9 In example 6.4.3 we saw that

$$\gamma_1(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \gamma_2(\theta) = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

give opposite orientations. Confirm (proposition 6.4.8) that $\det[\mathbf{D}(\gamma_2^{-1} \circ \gamma_1)] < 0$.

6.10 Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be coordinates in $\mathbb{C}^2$. Compute the integral of $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ over the part of the locus of equation $z_2 = z_1^k$ where $|z_1| < 1$, oriented by $\operatorname{sgn} dx_1 \wedge dy_1$.

6.11 For the following 1-forms, write down the corresponding vector field. Sketch the vector field. Describe a path over which the work of the 1-form would be small. Describe a path over which the work would be large.

a. $(x^2 + y^2)\, dz$, on $\mathbb{R}^3$ b. $y\, dx - x\, dy - z\, dz$, on $\mathbb{R}^3$

6.12 a. In $\mathbb{R}^2$, a vector field defines two 1-forms: the work and the flux. Show that they are related by formula

$$W_{\vec{F}}(\vec{\mathbf{v}}) = \Phi_{\vec{F}}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{\mathbf{v}} \right).$$

b. What does the transformation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ correspond to geometrically?

c. Can you explain why the work and the flux on $\mathbb{R}^2$ are related by the formula in part a?

6.13 Find the flux of the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y \\ -z \\ yz \end{bmatrix}$, through S, where S is the part of the cone $z = \sqrt{x^2 + y^2}$ where $x, y \geq 0$, $x^2 + y^2 \leq R$, and it is oriented by the inward-pointing normal (i.e., the flux measures the amount flowing into the cone).

6.14 Let S be the part of the surface of equation $z = \sin xy + 2$ where

$$x^2 + y^2 \le 1 \quad \text{and} \quad x \ge 0,$$

oriented by the upward-pointing normal. What is the flux of the vector field $\begin{bmatrix} 0 \\ 0 \\ x+y \end{bmatrix}$ through S?

6.15 Consider the manifold $S^3 \subset \mathbb{R}^4$ of equation $x_1^2+x_2^2+x_3^2+x_4^2 = 1$, oriented by $\Omega_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3) = \operatorname{sgn}\det[\mathbf{x}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3]$. Let X be the subset of S^3 where $x_4 \le 0$.

a. Show that X is a piece-with-boundary of S^3.

b. Find a basis for the tangent space $T_{\mathbf{x}}(\partial X)$ at the point $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \partial X$ which is direct for the boundary orientation.

6.16 a. Compute the derivative of $xy\, dz$ from the definition.

b. Compute the same derivative using the formulas given in theorem 6.7.2, stating clearly at each stage what property you are using.

6.17 a. Let $\varphi = xyz\, dy$. Compute from the definition the number

$$\mathbf{d}\varphi \left(P_{\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)} (\vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3) \right).$$

b. What is $\mathbf{d}\varphi$? Use your result to check the computation in part a.

6.18 Let $\vec{\mathbf{r}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be the radial vector field in $\mathbb{R}^n$.

Exercise 6.18 gives another way to derive equation 5.3.52.

a. Show that $\mathbf{d}(\Phi_{\vec{\mathbf{r}}}) = n(dx_1 \wedge \cdots \wedge dx_n)$.

b. Let $B_1^n(\mathbf{0})$ and S^{n-1} be the unit ball and the unit sphere in $\mathbb{R}^n$, the ball with the standard orientation and the sphere with the boundary orientation. Use Stokes's theorem to prove

$$\operatorname{vol}_n \left(B_1^n(\mathbf{0}) \right) = \frac{1}{n} \operatorname{vol}_{n-1}(S^{n-1}).$$

6.19 Using the formulas of theorem 6.8.3, prove the equations

$$\operatorname{curl}(\operatorname{grad} f) = \vec{\mathbf{0}} \quad \text{and} \quad \operatorname{div}(\operatorname{curl} \vec{F}) = 0$$

for any function f and any vector field $\vec{F}$ (of class at least C^2).

6.20 a. For what vector field $\vec{F}$ is the 1-form on $\mathbb{R}^3$

$$x^2\, dx + y^2 z\, dy + xy\, dz \quad \text{the work form field } W_{\vec{F}}?$$

b. Compute the exterior derivative of $x^2\, dx + y^2 z\, dy + xy\, dz$ using theorem 6.7.2. Show that it is the same as $\Phi_{\vec{\nabla} \times \vec{F}}$.

6.21 a. There is an exponent m such that

$$\vec{\nabla} \cdot (x^2 + y^2 + z^2)^m \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0; \quad \text{find it.}$$

*b. More generally, there is an exponent m (depending on n) such that the $(n-1)$-form $\Phi_{r^{2m}\vec{r}}$ has exterior derivative 0, when $\vec{r}$ is the vector field $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, and $r = |\vec{r}|$. Can you find it? (Start with $n = 1$ and $n = 2$.)

Exercise 6.21, part b: The subscript on Φ may be hard to read. It is $r^{2m}\vec{r}$.

6.22 a. Find the unique polynomial p such that $p(1) = 1$ and such that if

$$\omega = x\,dy \wedge dz - 2zp(y)\,dx \wedge dy + yp(y)\,dz \wedge dx,$$

then $\mathbf{d}\omega = dx \wedge dy \wedge dz$.

b. For this polynomial p, find the integral $\int_S \omega$, where S is that part of the sphere $x^2+y^2+z^2 = 1$ where $z \geq \sqrt{2}/2$, oriented by the outward-pointing normal.

6.23 a. Compute the exterior derivative of the 2-form

$$\varphi = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \Big(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy \Big).$$

b. Compute the integral of φ over the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the outward-pointing normal.

c. Compute the integral of φ over the boundary of the cube of side 4, centered at the origin, and oriented by the outward-pointing normal.

d. Can φ be written $\mathbf{d}\psi$ for some 1-form ψ on $\mathbb{R}^3 - \{\mathbf{0}\}$?

6.24 Let S be the surface of equation $z = 9 - y^2$, oriented by the upward-pointing normal.

a. Sketch the piece $X \subset S$ where $x \geq 0$, $z \geq 0$, and $y \geq x$, indicating carefully the boundary orientation.

b. Give a parametrization of X, being careful about the domain of the parametrizing map and whether it is orientation preserving.

c. Find the work of the vector field $\begin{bmatrix} 0 \\ xz \\ 0 \end{bmatrix}$ around the boundary of X.

Exercise 6.26: Equation 6.11.72 relates the coordinates x', y', z', t' of a frame of reference moving at speed v in the direction of the x-axis with respect to a frame with coordinates x, y, z, t. A particle at rest in the original frame has Faraday 2-form given by exercise 6.31; in the moving frame it is our required particle moving at constant speed. Therefore it is enough to write the form of exercise 6.31 in the new coordinates.

6.25 Let $U \subset \mathbb{R}^3$ be a subset bounded by a surface S, to which we will give the boundary orientation. What relation is there between the volume of U and the flux $\int_S \Phi_{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$?

6.26 Show that the electromagnetic field of a charge q moving in the direction of the x-axis at constant speed v is

$$\vec{\mathbf{E}} = \frac{q}{((\gamma x - \gamma vt)^2 + y^2 + z^2)^{3/2}} \begin{bmatrix} \gamma(x - vt) \\ y \\ z \end{bmatrix}$$

$$\vec{\mathbf{B}} = \frac{q}{((\gamma x - \gamma vt)^2 + y^2 + z^2)^{3/2}} \begin{bmatrix} 0 \\ -\gamma vz/c^2 \\ \gamma vz/c^2 \end{bmatrix},$$

where $\gamma = \sqrt{1 - v^2/c^2}$.

6.27 Let $\vec{F}$ be the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \\ 0 \end{bmatrix}$.

Suppose $D_2F_1 = D_1F_2$. Show that there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \vec{\nabla} f$.

6.28 Find a 1-form φ such that $\mathbf{d}\varphi = y\,dz \wedge dx - x\,dy \wedge dz$.

6.29 Let $U \subset \mathbb{R}^3$ be a 3-dimensional piece-with-boundary.

a. What does Stokes's theorem say about $\displaystyle\int_U \mathbf{d}\mathbb{F}$?

b. What does Stokes's theorem say about $\displaystyle\int_U \mathbf{d}\mathbb{M}$?

6.30 Let $S \subset \mathbb{R}^3$ be a 2-dimensional oriented piece-with-boundary, and let $I = [t_0, t_1]$ be a time interval.

a. Show that $V = S \times I$ is a 3-dimensional piece-with-boundary of $\mathbb{R}^4$.

b. What does Stokes's theorem say about $\displaystyle\int_V \mathbf{d}\mathbb{F}$?

c. What does Stokes's theorem say about $\displaystyle\int_V \mathbf{d}\mathbb{M}$?

Exercise 6.31: Note that

$$\mathbf{d}\left(\frac{1}{r}\right) = W_{\vec{\mathbf{r}}/r^3}.$$

This contrasts with example 6.12.1. There we had

$$\mathbf{d} \arctan \frac{y}{x} = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

But $\arctan \frac{y}{x}$ *is not* a well-defined function on $\mathbb{R}^2 - \{\mathbf{0}\}$, whereas $1/r$ is a well-defined function on $\mathbb{R}^3 - \{\mathbf{0}\}$.

Forms were defined to be integrands, so it is quite reasonable to define $\mathbf{d}\mathbb{F}$ via its integrals over oriented pieces-with-boundary. It is also quite reasonable to restrict to r-adapted pieces: pieces-with-boundary for which, locally near points where $r = 0$, the piece X is the graph of a map expressing t as a function of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Pieces that are not r-adapted intersect the locus $r = 0$ in exceptional ways, and it may be difficult to say what share of whatever nastiness is hiding there is carried by X. Such pieces are exceptional; by budging them arbitrarily little we can make them r-adapted.

****6.31** Let $\vec{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $r = |\vec{\mathbf{r}}| = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\mathbb{F} = qW_{\vec{\mathbf{r}}/r^3} \wedge c\,dt$$

is an electromagnetic field. What are the corresponding charges and currents?

This isn't a straightforward computation. For $\mathbb{F}$ to be an electromagnetic field, we must have $\mathbf{d}\mathbb{F} = 0$ everywhere, and it isn't clear what this means when $r = 0$. In example 6.12.1 we saw that there can be trouble hiding at points where a form is not defined. We deal with this as follows.

A 3-dimensional piece-with-boundary $X \subset \mathbb{R}^4$ will be called r-*adapted* if locally near points where $r = 0$ it represents t as a function of x, y, and z; in that case we write $X_\epsilon = X - \{r < \epsilon\}$ and we define

$$\partial_{inn} X_\epsilon \subset \partial X_\epsilon$$

to be the subset where $r = \epsilon$, with the boundary orientation. If φ is a 2-form on $\mathbb{R}^4 - \{r = 0\}$ and $X \subset \mathbb{R}^4$ is an r-adapted oriented 3-dimensional piece-with-boundary, we define

$$\int_X \mathbf{d}\varphi \overset{\text{def}}{=} \lim_{\epsilon \to 0} \left(\int_{X_\epsilon} \mathbf{d}\varphi - \int_{\partial_{inn} X_\epsilon} \varphi \right).$$

If φ is well defined where $r = 0$, then, by Stokes's theorem,

$$\int_X \mathbf{d}\varphi = \int_{X_\epsilon} \mathbf{d}\varphi + \int_{X - X_\epsilon} \mathbf{d}\varphi = \int_{X_\epsilon} \mathbf{d}\varphi - \int_{\partial_{inn} X_\epsilon} \varphi,$$

so the formula is correct in that case, and otherwise the boundary term captures whatever is hiding on $\{r = 0\}$.

6.32 Compute the integral $\displaystyle\int_S \Phi_{\vec{F}}$, where $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -x^2yz \\ y \\ (z^2-1)xy \end{bmatrix}$, and S is the part of the parabolic cylinder of equation $y = 9 - x^2$ where $y \geq 0$ and $0 \leq z \leq 1$, oriented by the transverse vector field $\vec{\mathbf{e}}_2$.