b. Use part a to compute $\mathbf{d}\Phi_{\vec{F}} P_{\begin{pmatrix}1\\1\\2\end{pmatrix}}(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3)$.

c. Compute it again, directly from the definition of the exterior derivative.

$$\begin{bmatrix} x^2y \\ -2yz \\ x^3y^2 \end{bmatrix} \qquad \begin{bmatrix} \sin xz \\ \cos yz \\ xyz \end{bmatrix}$$

Vector fields for exercise 6.8.11.

6.8.11 a. Compute the divergence and curl of the vector fields in the margin.

b. Compute them again, directly from the definition of the exterior derivative.

6.8.12 Show that

$$\operatorname{grad} \operatorname{div} \vec{\mathbf{F}} - \operatorname{curl} \operatorname{curl} \vec{\mathbf{F}} = \begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \end{bmatrix}.$$

6.9 The generalized Stokes's theorem

We worked hard to define the exterior derivative and to define orientation of manifolds and of boundaries. Now we are going to reap some rewards for our labor: we are going to see that there is a higher-dimensional analogue of the fundamental theorem of calculus, Stokes's theorem. It covers in one statement the four integral theorems of vector calculus, which are explored in section 6.10.

Recall the fundamental theorem of calculus:

Theorem 6.9.2 is probably the best tool mathematicians have for deducing global properties from local properties. It is a wonderful theorem.

It is often called the generalized Stokes's theorem, to distinguish it from the special case (surfaces in $\mathbb{R}^3$) also known as Stokes's theorem. Special cases of the generalized Stokes's theorem are discussed in section 6.10.

To lighten notation, in theorem 6.9.2 we write ∂X. However, we are actually integrating φ over $\partial_M^s X$, the smooth part of the boundary that sets off $X \subset M$ from M.

Theorem 6.9.1 (Fundamental theorem of calculus). *If f is a C^1 function on a neighborhood of $[a,b]$, then*

$$\int_a^b f'(t)\,dt = f(b) - f(a). \qquad 6.9.1$$

Restate this as

$$\int_{[a,b]} \mathbf{d}f = \int_{\partial[a,b]} f, \qquad 6.9.2$$

i.e., the integral of $\mathbf{d}f$ over the oriented interval $[a,b]$ is equal to the integral of f over the oriented boundary $+b-a$ of the interval. This is the case $k=n=1$ of theorem 6.9.2:

Theorem 6.9.2 (Generalized Stokes's theorem). *Let X be a compact piece-with-boundary of a k-dimensional oriented manifold $M \subset \mathbb{R}^n$. Give the boundary ∂X of X the boundary orientation, and let φ be a $(k-1)$-form defined on an open set containing X. Then*

$$\int_{\partial X} \varphi = \int_X \mathbf{d}\varphi. \qquad 6.9.3$$

This beautiful, short statement is the main result of the theory of forms. Note that the dimensions in equation 6.9.3 make sense: if X is k-dimensional, ∂X is $(k-1)$-dimensional, and if φ is a $(k-1)$-form, $\mathbf{d}\varphi$ is a k-form, so $\mathbf{d}\varphi$ can be integrated over X, and φ can be integrated over ∂X.

You apply Stokes's theorem every time you use antiderivatives to compute an integral: to compute the integral of the 1-form $f\,dx$ over the oriented line segment $[a,b]$, you begin by finding a function g such that $\mathbf{d}g = f\,dx$, and then say

$$\int_a^b f\,dx = \int_{[a,b]} \mathbf{d}g = \int_{\partial[a,b]} g = g(b) - g(a). \tag{6.9.4}$$

This isn't quite the way Stokes's theorem is usually used in higher dimensions, where "looking for antiderivatives" has a different flavor.

FIGURE 6.9.1.
Elie Cartan (1869–1951) formalized the theory of differential forms in the early twentieth century. Other names associated with the generalized Stokes's theorem include Henri Poincaré, Vito Volterra, and Luitzen Brouwer.

One of Cartan's four children, Henri, became a renowned mathematician; he died August 13, 2008, at the age of 104. Another, a physicist, was arrested by the Germans in 1942 and executed 15 months later.

Example 6.9.3 (Integrating over the boundary of a square). For instance, to compute the integral $\int_C x\,dy - y\,dx$, where C is the boundary of the square S described by the inequalities $|x|, |y| \le 1$, with the boundary orientation, one possibility is to parametrize the four sides of the square (being careful to get the orientations right), then to integrate $x\,dy - y\,dx$ over all four sides and add. Another possibility is to apply Stokes's theorem:

$$\int_C x\,dy - y\,dx = \int_S \mathbf{d}(x\,dy - y\,dx) = \int_S 2\,dx \wedge dy = \int_S 2|dx\,dy| = 8. \tag{6.9.5}$$

(The square S has sidelength 2, so its area is 4.) $\triangle$

What is the integral over C of $x\,dy + y\,dx$? Check below.[13]

Example 6.9.4 (Integrating over the boundary of a cube). Let us integrate the 2-form

$$\varphi = (x - y^2 + z^3)\,(dy \wedge dz + dx \wedge dz + dx \wedge dy) \tag{6.9.6}$$

over the boundary of the cube C_a given by $0 \le x, y, z \le a$.

It is quite possible to do this directly, parametrizing all six faces of the cube, but Stokes's theorem simplifies things substantially.

Computing the exterior derivative of φ gives

$$\begin{aligned}\mathbf{d}\varphi &= dx \wedge dy \wedge dz - 2y\,dy \wedge dx \wedge dz + 3z^2\,dz \wedge dx \wedge dy \\ &= (1 + 2y + 3z^2)\,dx \wedge dy \wedge dz,\end{aligned} \tag{6.9.7}$$

so we have

$$\begin{aligned}\int_{\partial C_a} \varphi &= \int_{C_a} (1 + 2y + 3z^2)\,dx \wedge dy \wedge dz \\ &= \int_0^a \int_0^a \int_0^a (1 + 2y + 3z^2)\,dx\,dy\,dz \\ &= a^2\big([x]_0^a + [y^2]_0^a + [z^3]_0^a\big) = a^2\big(a + a^2 + a^3\big). \quad \triangle\end{aligned} \tag{6.9.8}$$

[13] $\mathbf{d}(x\,dy + y\,dx) = dx \wedge dy + dy \wedge dx = 0$, so the integral is 0.