

that C is an uncountable set. (*Hint:* For instance, the number written as $.02220000022202002222\dots$ in base 3 is in it.)

- b. Show that C is a pavable set, with one-dimensional volume 0.
- c. Show that the only dimension in which C can have volume different from 0 or infinity is $\ln 2/\ln 3$.

5.5.2 This time let the set C be obtained from the unit interval by omitting the open middle $1/n$ th, then the open middle $1/n$ of each of the remaining intervals, then the open middle $1/n$ th of the remaining intervals, etc. (When n is even, this means omitting an open interval equivalent to $1/n$ th of the unit interval, leaving equal amounts on both sides, and so on.)

- a. Show that C is a pavable set, with one-dimensional volume 0.
- b. What is the only dimension in which C can have volume different from 0 or infinity? What is this dimension when $n = 2$?

5.6 REVIEW EXERCISES FOR CHAPTER 5

5.1 Verify that equation 5.3.31 parametrizes the torus obtained by rotating around the z -axis the circle of radius r in the (x, z) -plane that is centered at $x = R, z = 0$.

5.2 Let $f : [a, b] \rightarrow \mathbb{R}$ be a smooth positive function. Find a parametrization for the surface of equation $\frac{x^2}{A^2} + \frac{y^2}{B^2} = (f(z))^2$.

5.3 For what values of α does the spiral $\begin{pmatrix} r(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} 1/t^\alpha \\ t \end{pmatrix}, \alpha > 0$ between $t = 1$ and $t = \infty$ have finite length?

5.4 Compute the area of the graph of the function $f\left(\frac{x}{y}\right) = \frac{2}{3}(x^{3/2} + y^{3/2})$ above the region $0 \leq x, y \leq 1$.

5.5 Let $f(x)$ be a positive C^1 function of $x \in [a, b]$.

Exercise 5.5, part b: The answer should be in the form of a one-dimensional integral.

- a. Find a parametrization of the surface in \mathbb{R}^3 obtained by rotating the graph of f around the x -axis.
- b. What is the area of this surface?

5.6 Let $w_{n+1}(r) = \text{vol}_{n+1}(B_r^{n+1}(\mathbf{0}))$ be the $(n+1)$ -dimensional volume of the ball of radius r in \mathbb{R}^{n+1} , and let $v_n(r) = \text{vol}_n(S_r^n)$ be the n -dimensional volume of the sphere of radius r in \mathbb{R}^{n+1} .

- a. Show that $w'_{n+1}(r) = v_n(r)$.
- b. Show that $v_n(r) = r^n v_n(1)$.
- c. Derive equation 5.3.52, using $w_{n+1}(1) = \int_0^1 w'_{n+1}(r) dr$.

The *total curvature* of a surface is defined in exercise 5.3.11.

5.7 Let H be the helicoid of equation $y \cos z = x \sin z$ (see example 3.8.13). What is the total curvature of the part of H with $0 \leq z \leq a$?

5.8 For $z \in \mathbb{C}$, the function $\cos z$ is by definition $\cos z = \frac{e^{iz} + e^{-iz}}{2}$.

- a. If $z = x + iy$, write the real and imaginary parts of $\cos z$ in terms of x, y .
- b. What is the area of the part of the graph of $\cos z$ where $-\pi \leq x \leq \pi$ and $-1 \leq y \leq 1$?

5.9 Let the set C be obtained from the unit interval $[0, 1]$ by omitting the open middle 1/5th, then the open middle fifth of each remaining interval, then the open middle fifth of each remaining interval, etc.

a. Show that an alternative description of C is that it is the set of points that can be written in base 5 without using the digit 2. Use this to show that C is an uncountable set.

b. Show that C is a pavable set, with one-dimensional volume 0.

c. What is the only dimension in which C can have volume different from 0 or infinity?

Exercise 5.10 is inspired by a proposition in *Functional Analysis, Volume 1: A Gentle Introduction*, by Dzung Minh Ha (Matrix Editions, 2006).

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{x}_0}, \quad \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}}_{\vec{x}_1}, \quad \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}}_{\vec{x}_2}$$

Vectors for exercise 5.10, part b

5.10 Let $\vec{x}_0, \vec{x}_1, \dots, \vec{x}_k$ be vectors in \mathbb{R}^n , and let $M \subset \mathbb{R}^n$ be the subspace spanned by $\vec{x}_1, \dots, \vec{x}_k$. Let G be the $k \times k$ matrix $G = [\vec{x}_1 \dots \vec{x}_k]^\top [\vec{x}_1 \dots \vec{x}_k]$ and let G^+ be the $(k+1) \times (k+1)$ matrix $G^+ = [\vec{x}_0 \vec{x}_1 \dots \vec{x}_k]^\top [\vec{x}_0 \vec{x}_1 \dots \vec{x}_k]$. The distance $d(\mathbf{x}, M)$ is by definition $d(\mathbf{x}, M) = \inf_{\mathbf{y} \in M} \|\mathbf{x} - \mathbf{y}\|$.

a. Show that

$$\left(d(\mathbf{x}, M)\right)^2 = \frac{\det G^+}{\det G}.$$

b. What is the distance between \vec{x}_0 and the plane spanned by \vec{x}_1 and \vec{x}_2 (as defined in the margin)?