5.3.18 A gas has density C/r, where $r = \sqrt{x^2+y^2+z^2}$. If $0 < a < b$, what is the mass of the gas between the concentric spheres $r = a$ and $r = b$?

5.3.19 Justify the last line of equation 5.3.51 by computing the integral in the third line.

5.3.20 Let $M_1(n,m)$ be the space of $n \times m$ matrices of rank 1. What is the three-dimensional volume of the part of $M_1(2,2)$ made up of matrices A with $|A| \le R$, for $R > 0$?

5.3.21 What is the area of the surface in $\mathbb{C}^3$ parametrized by

$$\gamma(z) = \begin{pmatrix} z^p \\ z^q \\ z^r \end{pmatrix}, \quad z \in \mathbb{C},\ |z| \le 1\,?$$

5.4 Integration and curvature

In this section we prove three theorems relating curvature to integration: Gauss's *remarkable theorem*, which explains Gaussian curvature; theorem 5.4.4, which explains mean curvature; and theorem 5.4.6, which relates curvature to the Gauss map.

Gauss, writing in Latin, called this result the "remarkable theorem: If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged." Gauss's own proof is a complicated and unmotivated computation, the kind of thing that led Jacobi to liken Gauss to the *fox who erases his tracks in the sand with his tail*; see page 4.

The Latin word "egregius" (literally, *not of the common herd*) is translated as *admirable, excellent, extraordinary, distinguished, …* . This laudatory meaning persisted in English through the 19th century (Thackeray wrote of "some one splendid and egregious"), but the word was also used in its current meaning, "remarkably bad"; in Shakespeare's *Cymbeline*, a character declares himself to be an "egregious murderer".

An *isometry* is a map that preserves lengths of all curves.

A proof of Gauss's remarkable theorem

Perhaps the most famous theorem of differential geometry is Gauss's *Theorema Egregium* (Latin for *remarkable theorem*), which asserts that the Gaussian curvature of a surface is "intrinsic": it can be computed from lengths and angles measured in the surface.

It is remarkable because the Gaussian curvature is the product of two quantities that themselves are *extrinsic*: they depend on how the surface is embedded in $\mathbb{R}^3$.

We saw in section 3.8 that in "best coordinates", a surface $S \subset \mathbb{R}^3$ is locally near every $\mathbf{p} \in S$ the graph of a map from its tangent space $T_\mathbf{p}S$ to its normal line N_p. The Taylor polynomial of this map starts with quadratic terms:

$$Z = f\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2}(A_{2,0}X^2 + 2A_{1,1}XY + A_{0,2}Y^2) + o(X^2+Y^2). \qquad 5.4.1$$

(See exercise 3.3.15.)

The above is *extrinsic*: if $\varphi : S \to S'$ is an isometry between embedded surfaces, the coefficients $A_{2,0}$, $A_{1,1}$, and $A_{0,2}$ of the two embedded surfaces will usually be different. Yet Gauss showed that the Gaussian curvature

$$K(\mathbf{p}) = A_{2,0}A_{0,2} - A_{1,1}^2. \qquad 5.4.2$$

is intrinsic.

We said in section 3.8 that one way to express the Theorema Egregium is as theorem 3.8.9, repeated below as theorem 5.4.1.

The other proofs of theorem 5.4.1 that we know all involve either the Christoffel symbol $\Gamma_{1,2}$ (hence the Levi-Civita connection associated to the Riemann metric on S inherited from the embedding), or the Jacobi second variation equation, which describes how geodesics on S spread apart.

These tools are crucial to any serious study of differential geometry, but they are beyond the scope of this book. The present proof avoids these more advanced tools, using instead techniques and concepts developed in chapters 3, 4, and 5, as well as rules for "little o" and "big O" given in appendix A.11.

Theorem 5.4.1. *Let $D_r(\mathbf{p})$ be the set of all points $\mathbf{q}$ in a surface $S \subset \mathbb{R}^3$ such that there exists a curve of length $\leq r$ in S joining $\mathbf{p}$ to $\mathbf{q}$. Then*

$$\operatorname{Area}(D_r(\mathbf{p})) = \pi r^2 - \frac{\pi K(\mathbf{p})}{12} r^4 + o(r^4). \qquad 5.4.3$$

Since the area of $D_r(\mathbf{p})$, the disc of radius r around $\mathbf{p}$, is obviously an "intrinsic" function and it determines the curvature $K(\mathbf{p})$, the Theorema Egregium follows.

Surfaces and graphs

Since theorem 5.4.1 is local, we may assume that $\mathbf{p} = \mathbf{0}$ and that S is the graph of a smooth function defined near the origin of $\mathbb{R}^2$, whose Taylor polynomial starts with quadratic terms, as in equation 5.4.1. These quadratic terms are a quadratic form Q_A on $\mathbb{R}^2$, where $A = \begin{bmatrix} A_{2,0} & A_{1,1} \\ A_{1,1} & A_{0,2} \end{bmatrix}$. Since A is symmetric, by the spectral theorem (theorem 3.7.14) there exists an orthonormal basis in which the matrix A is diagonal. Thus we may assume that our surface is the graph of a function

Equation 5.4.4: To lighten notation, we replace $A_{2,0}$ by a and $A_{0,2}$ by b.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}(ax^2 + by^2) + o(x^2 + y^2), \qquad 5.4.4$$

so that the Gaussian curvature at the origin is $K(\mathbf{0}) = ab$. Thus to prove theorem 5.4.1, we need to show that

$$\operatorname{Area}(D_r(\mathbf{0})) = \pi r^2 - \frac{ab\pi}{12} r^4 + o(r^4). \qquad 5.4.5$$

We know from definition 5.3.1 how to compute the volume of a manifold known by a parametrization. The first step is to parametrize the surface S. We will use the "radial parametrization"

Equation 5.4.6 is an example of parametrizing a surface as a graph; see example 5.2.5.

$$g\begin{pmatrix} \rho \\ \theta \end{pmatrix} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ f\begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ \frac{1}{2}\rho^2(a\cos^2\theta + b\sin^2\theta) + o(\rho^2) \end{pmatrix} \qquad 5.4.6$$

However, to apply definition 5.3.1, we need to find a subset $U_r \subset \mathbb{R}^2$ such that $g(U_r)$ and $D_r(0)$ have the same area, to within terms in $o(r^4)$, so that we can write

We speak of curves that almost minimize distance because proving that geodesics (curves that actually minimize distances) exist in S is beyond the scope of this book.

$$\operatorname{Area} D_r(0) = \int_{U_r} \sqrt{\det\left[\mathbf{D}g\begin{pmatrix} \rho \\ \theta \end{pmatrix}\right]^\top \left[\mathbf{D}g\begin{pmatrix} \rho \\ \theta \end{pmatrix}\right]}\, |d\rho\, d\theta| + o(r^4). \qquad 5.4.7$$

We will begin by finding a parametrized curve $\widetilde{\gamma}$ in S such that the distance along the curve from $\mathbf{0}$ to another point in S is sufficiently close to minimal so that our error is in $o(r^4)$. We can think of the arc $\widetilde{\gamma}([0, r])$