4.11.12 For what values of a and b is $f\begin{pmatrix} x \\ y \end{pmatrix} = x^a y^b$ integrable over the region $x \geq 0$, $y \geq 0$, $xy \leq 1$? (This isn't too hard to do directly, but using the change of variables formula given in theorem 4.11.21 makes it easier.)

4.11.13 a. Show that the three sequences of functions in example 4.11.3 do not converge uniformly.

*b. Show that the sequence of polynomials of degree $\leq m$

$$p_k(x) = a_{0,k} + a_{1,k}x + \cdots + a_{m,k}x^m$$

does not converge uniformly on $\mathbb{R}$, unless the sequence $a_{i,k}$ is eventually constant for all $i > 0$ and the sequence $a_{0,k}$ converges.

c. Show that if the sequences $a_{i,k}$ converge for each $i \leq m$, and A is a bounded set, then the sequence $p_k \mathbf{1}_A$ converges uniformly.

4.11.14 For the first two sequences of functions in example 4.11.3, show that

$$\lim_{k\to\infty} \lim_{R\to\infty} \int_{\mathbb{R}} [f_k]_R(x)\, dx \neq \lim_{R\to\infty} \lim_{k\to\infty} \int_{\mathbb{R}} [f_k]_R(x)\, dx.$$

4.11.15 Show that the series $\displaystyle\sum_{i=0}^{\infty} \int_{1/2^{i+1}}^{1/2^i} |\ln x|\, dx$ of example 4.11.12 converges.

4.11.16 Show that the integral $\int_0^\infty \frac{\sin x}{x}\, dx$ of equation 4.11.50 is equal to the sum of the series

$$\sum_{k=1}^{\infty} \left(\int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x}\, dx \right),$$

and that this series converges.

4.11.17 Make the change of variables $u = 1/x$ in the integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$. Does the resulting integral exist as an improper integral, as described in example 4.11.13?

Exercise 4.11.18: You will need the dominated convergence theorem (theorem 4.11.4) to prove this.

Part b: "Except at 0" means "except at 0 of P_k," i.e., the zero polynomial.

4.11.18 Let P_k be the space of polynomials of degree at most k. Consider the function $F : P_k \to \mathbb{R}$ given by $p \mapsto \int_0^1 |p(x)|\, dx$.

a. Compute F when $k = 1$ and $k = 2$, i.e., evaluate the integrals $\int_0^1 |a + bx|\, dx$ and $\int_0^1 |a + bx + cx^2|\, dx$ (the second one is hard).

b. Show that F is differentiable except at 0. Compute the derivative.

*c. Show that if p has only simple roots between 0 and 1, then F is twice differentiable at p.

4.12 Review exercises for chapter 4

Exercise 4.1:

$$U_N(\mathbf{1}_C) = L_N(\mathbf{1}_C).$$

4.1 Show that if $C \in \mathcal{D}(\mathbb{R}^m)$, then $\mathbf{1}_C$ is integrable.

4.2 An integrand should take a piece of the domain and return a number, in such a way that if we decompose a domain into little pieces, evaluate the integrand on the pieces, and add, the sums should have a limit as the decomposition becomes infinitely fine (and the limit should not depend on how the domain is decomposed). What will happen if we break up $[0,1]^2$ into rectangles defined by $a < x < b$ and $c < y < d$ and assign one of the numbers below to each rectangle?

a. $|ac - bd|$ b. $(ad - bc)^2$.

4.3 Let A be an $n \times n$ matrix of integers, viewed as a map $\mathbb{Z}^n \to \mathbb{Z}^n$. Which of the following are true?

1. $\ker A = 0 \implies A$ is onto.
2. A onto $\implies \ker A = 0$.
3. $\det A \neq 0 \implies \ker A = 0$.
4. $\det A \neq 0 \implies A$ is onto.

4.4 Which elementary matrices are permutation matrices? Describe the corresponding permutations.

4.5 Evaluate $\displaystyle \lim_{N\to\infty} \frac{1}{N^2} \sum_{k=1}^{N} \sum_{l=1}^{2N} e^{\frac{k+l}{N}}$.

4.6 What are the expectation, variance, and standard deviation of the random variable $f(x) = x$, for the following probability densities.

$$\text{a. } \mu(x) = e^{-x}\mathbf{1}_{[0,\infty]} \qquad \text{b. } \mu(x) = \frac{1}{x^2}\mathbf{1}_{[0,\infty]}$$

4.7 Let A and B be two disjoint bodies, with densities μ_1 and μ_2 and masses $M(A)$ and $M(B)$. Set $C = A \cup B$. Show that the center of gravity of C is

$$\overline{\mathbf{x}}(C) = \frac{M(A)\overline{\mathbf{x}}(A) + M(B)\overline{\mathbf{x}}(B)}{M(A) + M(B)}.$$

4.8 Prove corollary 4.3.11.

4.9 Let X be a subset of $\mathbb{R}^n$ such that for any $\epsilon > 0$, there exists a sequence of pavable sets $B_i,\ i = 1, 2, \dots$ satisfying $X \subset \bigcup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} \mathrm{vol}_n(B_i) < \epsilon$. Show that X has measure 0.

4.10 Give an explicit upper bound (in terms of N) for the number of cubes in $\mathcal{D}_N(\mathbb{R}^3)$ needed to cover the unit sphere $S^2 \subset \mathbb{R}^3$, such that the volume of the cubes tends to 0 as N tends to infinity.

4.11 Write each of the following double integrals as iterated integrals in two ways, and compute them:

a. The integral of $\sin(x + y)$ over the region $x^2 < y < 2$

b. The integral of $x^2 + y^2$ over the region $1 \le |x|, |y| \le 2$

4.12 Compute the integral of the function z over the region R described by the inequalities $x > 0,\ y > 0,\ z > 0,\ x + 2y + 3z < 1$.

4.13 a. If $f\begin{pmatrix} x \\ y \end{pmatrix} = a + bx + cy$, what are

$$\int_0^1 \int_0^2 f\begin{pmatrix} x \\ y \end{pmatrix} |dx\, dy| \quad \text{and} \quad \int_0^1 \int_0^2 \left(f\begin{pmatrix} x \\ y \end{pmatrix} \right)^2 |dx\, dy|?$$

b. Let f be as in part a. What is the minimum of $\int_0^1 \int_0^2 \left(f\begin{pmatrix} x \\ y \end{pmatrix} \right)^2 |dx\, dy|$ among all functions f such that

$$\int_0^1 \int_0^2 f\begin{pmatrix} x \\ y \end{pmatrix} |dx\, dy| = 1?$$

4.14 What is the z-coordinate of the center of gravity of the region

$$\frac{x^2}{(z^3-1)^2}+\frac{y^2}{(z^3+1)^2}\leq 1, \quad 0\leq z\leq 1\,?$$

4.15 Show that there exist c and u such that when f is any polynomial of degree $d \leq 3$,

$$\int_{-1}^{1} f(x)\frac{1}{\sqrt{1-x^2}}\,dx = c\Big(f(u)+f(-u)\Big).$$

4.16 Repeat exercise 4.6.4, parts a–d, but this time for the weight e^{-x^2} and limits of integration $-\infty$ to ∞; i.e., find points x_i and w_i such that

$$\int_{-\infty}^{\infty} p(x)e^{-x^2}\,dx = \sum_{i=0}^{k} w_i p(x_i)$$

is true for all polynomials of degree $\leq 2k-1$.

e. For each of the five values of m in part d, approximate

$$\int_{-\infty}^{\infty} e^{-x^2}\cos x\,dx \quad \text{and} \quad \int_{-\infty}^{\infty}\frac{e^{-x^2}}{1+x^2}\,dx.$$

Compare the approximations with the exact values.

4.17 Check part 3 of theorem 4.8.14 when $A=\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$; that is, show that $[\mathbf{D}\det(A)]B = \det A\ \mathrm{tr}(A^{-1}B)$.

Exercise 4.18: Start with corollary 4.8.16, and set

$$C = P,\ D = AP^{-1}.$$

This proves the formula when C is invertible. Complete the proof by showing that if C_n is a sequence of matrices converging to C, and $\mathrm{tr}(C_nD)=\mathrm{tr}(DC_n)$ for all n, then $\mathrm{tr}(CD)=\mathrm{tr}(DC)$.

4.18 Show that if A and B are $n\times n$ matrices, then $\mathrm{tr}(AB)=\mathrm{tr}(BA)$.

4.19 What is the n-dimensional volume of the region

$$\{\,\mathbf{x}\in\mathbb{R}^n \mid x_i\geq 0 \text{ for all } i=1,\dots,n \text{ and } x_1+2x_2+\cdots+nx_n\leq n\,\}?$$

4.20 a. Find an expression for the area of the parallelogram spanned by $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, in terms of $|\vec{\mathbf{v}}_1|, |\vec{\mathbf{v}}_2|$, and $|\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_2|$.

b. Prove Heron's formula: A triangle with sides of length a, b, c, has area

$$\sqrt{p\,(p-a)(p-b)(p-c)}\ , \quad \text{where} \quad p=\frac{a+b+c}{2}.$$

4.21 a. Sketch the curve in the plane given in polar coordinates by the equation $r = 1+\sin\theta, \quad 0\leq\theta\leq 2\pi$.

b. Find the area that the curve encloses.

4.22 A semicircle of radius R has density $\rho\begin{pmatrix}x\\ y\end{pmatrix}=m(x^2+y^2)$ proportional to the square of the distance to the center. What is its mass?

4.23 a. Let Q be the part of the unit ball $x^2+y^2+z^2\leq 1$ where $x,y,z\geq 0$. Using spherical coordinates, set up $\int_Q (x+y+z)\ |d^3\mathbf{x}|$ as an iterated integral.

b. Compute the integral.

4.24 Let $\mathbf{x}\in\mathbb{R}^n$. For what values of $p\in\mathbb{R}$ does $\int_{B_1(\mathbf{0})}|\mathbf{x}|^p\,|d^n\mathbf{x}|$ exist as a Lebesgue integral? (The answer depends on n.)

4.25 Let A be the region defined the inequalities $x^2+y^2\leq z\leq 1$. What is the center of gravity of A?

4.26 In this exercise we will show that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. This function is not Lebesgue integrable, and the integral should be understood as

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{a\to\infty}\int_0^a \frac{\sin x}{x}\,dx.$$

a. Show that for all $0 < a < b < \infty$,

$$\int_a^b \left(\int_0^\infty e^{-px}\sin x\,dx\right)dp = \int_0^\infty \left(\int_a^b e^{-px}\sin x\,dp\right)dx.$$

b. Use part a to show

$$\arctan b - \arctan a = \int_0^\infty \frac{(e^{-ax}-e^{-bx})\sin x}{x}\,dx.$$

c. Why does theorem 4.11.4 not imply that

$$\lim_{a\to 0}\lim_{b\to\infty}\int_0^\infty \frac{(e^{-ax}-e^{-bx})\sin x}{x}\,dx = \int_0^\infty \frac{\sin x}{x}\,dx\,? \tag{1}$$

Part d: Remember that the next omitted term is a bound for the error for each partial sum.

d. Prove that equation (1) is true anyway. The following lemma is the key: If $c_n(t) > 0$ are monotone increasing functions of t, with $\lim_{t\to\infty} c_n(t) = C_n$, and decreasing as a function of n for each fixed t, tending to 0, then

$$\lim_{t\to\infty}\sum_{n=1}^\infty (-1)^n c_n(t) = \sum_{n=1}^\infty (-1)^n C_n.$$

e. Write

$$\int_0^\infty \frac{(e^{-ax}-e^{-bx})\sin x}{x}\,dx = \sum_{n=0}^\infty \int_{k\pi}^{(k+1)\pi} (-1)^k\frac{(e^{-ax}-e^{-bx})|\sin x|}{x}\,dx,$$

and use part d to prove the equation $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

4.27 Let $a_1, a_2, \dots$ be a list of the rationals in $[0,1]$. Consider

$$f(x) = \sum_{k=1}^\infty \frac{1}{2^k}\frac{1}{\sqrt{|x-a_k|}}.$$

a. Show that f is L-integrable on $[0,1]$.

b. Show that the series converges for all x except x on a set of measure 0.

*c. Find an x for which the series converges.

Exercise 4.27, part c: This depends on the order chosen.

4.28 What are the expectation, variance, and standard deviation of the function (random variable) $f(x) = x$ for the probability density

$$\mu(x) = \frac{1}{x^2}\mathbf{1}_{[0,\infty]},$$

where the sample space is all of $\mathbb{R}$?

4.29 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(u) = au + b\bar{u}$, where we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the standard way. Show that

$$\det T = |a|^2 - |b|^2 \quad\text{and}\quad \|T\| = |a| + |b|.$$

Hint for part a of exercise 4.30: Show that if $g(\mathbf{x}_0) > f(\mathbf{x}_0)$ and $g - f$ is continuous at $\mathbf{x}_0$, then $\int g(\mathbf{x})|d^n\mathbf{x}| > \int f(\mathbf{x})|d^n\mathbf{x}|$. Then apply theorem 4.4.6.

4.30 a. Show that if f, g are R-integrable functions on $\mathbb{R}^n$ with $g \ge f$, and $\int f(\mathbf{x})|d^n\mathbf{x}| = \int g(\mathbf{x})|d^n\mathbf{x}|$, then $\{\, \mathbf{x} \mid f(\mathbf{x}) \ne g)\mathbf{x})\,\}$ has measure 0.

b. Show that the statement of part a is false if you replace "measure" by "volume".