b. Show that $\alpha(A+B) = \alpha A + \alpha B$, $1A = A$, and $(\alpha\beta)A = \alpha(\beta A)$, for all $\alpha, \beta \in \mathbb{R}$.

c. Is $\mathcal{P}^*(V)$ a vector space with these operations?

*d. Does $\mathcal{P}^*(V)$ have subsets that are vector spaces with these operations?

Exercise 2.34: The answer to part d depends on whether you choose the neutral element for addition to be $\{0\}$ or allow it to be whatever is appropriate for the particular subset you are looking at. In the latter case you might land on *quotient spaces*.

***2.35** This exercise gives a proof of *Bezout's theorem*. Let p_1 and p_2 be polynomials of degree k_1 and k_2 respectively, and consider the mapping

$$T\colon (q_1, q_2) \to p_1 q_1 + p_2 q_2,$$

where q_1 and q_2 are polynomials of degree at most $k_2 - 1$ and $k_1 - 1$ respectively, so that $p_1q_1 + p_2q_2$ is of degree $\le k_1 + k_2 - 1$. Note that the space of such (q_1, q_2) is of dimension $k_1 + k_2$, and the space of polynomials of degree at most $k_1 + k_2 - 1$ is also of dimension $k_1 + k_2$.

a. Show that $\ker T = \{0\}$ if and only if p_1 and p_2 are *relatively prime*.

b. Use corollary 2.5.10 to prove *Bezout's identity*: if p_1, p_2 are relatively prime, then there exist unique q_1 and q_2 of degree at most $k_2 - 1$ and $k_1 - 1$ such that $p_1q_1 + p_2q_2 = 1$.

Exercise 2.35, part a: It may be easier to work over the complex numbers.

Relatively prime: with no common factors.

****2.36** Let A be an $n \times n$ diagonal matrix: $A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, and suppose that one of the diagonal entries, say λ_k, satisfies $\sup_{k \ne j} |\lambda_k - \lambda_j| \ge m > 0$ for some number m. Let B be an $n \times n$ matrix. Find a number R, depending on m, such that if $|B| < R$, then Newton's method will converge if it is used to solve

$$(A+B)\mathbf{x} = \mu\mathbf{x}, \quad \text{for } \mathbf{x} \text{ satisfying } |\mathbf{x}|^2 = 1,$$

starting at $\mathbf{x}_0 = \vec{\mathbf{e}}_k$, $\mu_0 = \lambda_k$.

2.37 Find a bound on $a^2 + b^2$ such that Newton's method to solve

$$\begin{aligned} x^3 + x - 3y^2 &= a \\ x^5 + x^2y^3 - y &= b \end{aligned}$$

starting at $x = 0$, $y = 0$, is sure to converge to a solution.

Hint for exercise 2.37: set

$$\begin{aligned} z_1 &= x^2 \\ z_2 &= y^2 \\ z_3 &= z_1^2 \end{aligned}$$

2.38 Let a sequence of integers $a_0, a_1, a_2, \dots$ be defined inductively by

$$a_0 = 1, a_1 = 0, a_{n+2} = 2a_{n+1} + a_n \text{ for } n \ge 2.$$

a. Find a matrix M such that

$$\begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix} = M \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}.$$

Express $\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ in terms of powers of M.

b. Find a linear relation between I, M, M^2, and use it to find the eigenvalues of M.

c. Find a matrix P such that $P^{-1}MP$ is diagonal.

d. Compute M^n in terms of powers of numbers. Use the result to find a formula for a_n.