2.6.7 Let V be the vector space of C^1 functions on $(0,1)$. Which of the following are subspaces of V?

a. $\{ f \in V \mid f(x) = f'(x) + 1 \}$ b. $\{ f \in V \mid f(x) = xf'(x) \}$

c. $\left\{ f \in V \;\middle|\; f(x) = (f'(x))^2 \right\}$

Exercise 2.6.8: By "identified to $\mathbb{R}^3$ via the coefficients" we mean that

$$p(x) = a + bx + cx^2 \in P_2$$

is identified to

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Part c: The pattern should become clear after the first three.

2.6.8 Let P_2 be the space of polynomials of degree at most two, identified to $\mathbb{R}^3$ via the coefficients. Consider the mapping $T : P_2 \to P_2$ given by

$$T(p)(x) = (x^2+1)p''(x) - xp'(x) + 2p(x).$$

a. Verify that T is linear, i.e., that $T(ap_1 + bp_2) = aT(p_1) + bT(p_2)$.

b. Choose the basis of P_2 consisting of the polynomials $p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = x^2$. Denote by $\Phi_{\{p\}} : \mathbb{R}^3 \to P_2$ the corresponding concrete-to-abstract linear transformation. Show that the matrix of

$$\Phi^{-1}_{\{p\}} \circ T \circ \Phi_{\{p\}} \quad \text{is} \quad \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

c. Using the basis $1, x, x^2, \dots, x^n$, compute the matrices of the same differential operator T, viewed first as an operator from P_3 to P_3, then from P_4 to P_4, $\dots$, P_n to P_n (polynomials of degree at most $3, 4, \dots, n$).

Exercise 2.6.9 says that any linearly independent set can be extended to form a basis. In French treatments of linear algebra, this is called the *theorem of the incomplete basis*; it plus induction can be used to prove all the theorems of linear algebra in chapter 2.

2.6.9 a. Let V be a finite-dimensional vector space, and let $\underline{\mathbf{v}}_1, \dots, \underline{\mathbf{v}}_k \in V$ be linearly independent vectors. Show that there exist $\underline{\mathbf{v}}_{k+1}, \dots, \underline{\mathbf{v}}_n \in V$ such that $\{\underline{\mathbf{v}}\}_1, \dots, \underline{\mathbf{v}}_n$ is a basis of V.

b. Let V be a finite-dimensional vector space, and let $\underline{\mathbf{v}}_1, \dots, \underline{\mathbf{v}}_k \in V$ be a set of vectors that spans V. Show that there exists a subset $i_1, i_2, \dots, i_m$ of $\{1, 2, \dots, k\}$ such that $\underline{\mathbf{v}}_{i_1}, \dots, \underline{\mathbf{v}}_{i_m}$ is a basis of V.

2.6.10 Let $L_A, R_A : \text{Mat}\,(n,n) \to \text{Mat}\,(n,n)$ be the transformations

$$L_A,\ R_A : \text{Mat}\,(n,n) \to \text{Mat}\,(n,n) \qquad \text{given by}$$
$$L_A(B) = AB, \quad R_A(B) = BA.$$

What are $|L_A|$ and $|R_A|$ in terms of $|A|$ and $|B|$?

2.6.11 Let $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$. What is the dimension of the span of A, B, AB, and BA, in terms of a and b?

2.7 Eigenvectors and eigenvalues

When Werner Heisenberg discovered 'matrix' mechanics in 1925, he didn't know what a matrix was (Max Born had to tell him), and neither Heisenberg nor Born knew what to make of the appearance of matrices in the context of the atom. (David Hilbert is reported to have told them to go look for a differential equation with the same eigenvalues, if that would make them happier. They did not follow

Hilbert's well-meant advice and thereby may have missed discovering the Schrödinger wave equation.)—M. R. Schroeder, *Mathematical Intelligencer*, Vol. 7, No. 4

In section 2.6 we discussed the change of basis matrix, but never said why one would want to change bases. Of course, it is because a problem is easier in a different basis. Most often this comes down to some problem being easier in an *eigenbasis*: a basis of *eigenvectors*. Before defining the terms, let's give an example.

Example 2.7.1 (Fibonacci numbers). *Fibonacci numbers* are the numbers $1, 1, 2, 3, 5, 8, 13, \dots$ defined by $a_0 = a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$. We propose to prove the formula

$$a_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n. \qquad 2.7.1$$

We can analyze equation 2.7.1 to estimate how large a_n is. Note that for large n, the second term in equation 2.7.1 is negligible, since

$$\frac{1-\sqrt{5}}{2} \approx -0.618.$$

For instance, $(\frac{1-\sqrt{5}}{2})^{1000}$ starts with at least 200 zeros after the decimal point.

But the first term grows exponentially, since

$$\frac{1+\sqrt{5}}{2} \approx 1.618.$$

Assume $n = 1000$. Using logarithms base 10 to evaluate the first term, we see that

$$\begin{aligned}\log_{10} a_{1000} &\approx \log_{10}\frac{5+\sqrt{5}}{10} \\ &\quad + \left(1000 \times \log_{10}\frac{1+\sqrt{5}}{2}\right) \\ &\approx -.1405 + 1000 \times .20899 \\ &\approx 208.85,\end{aligned}$$

which means that a_{1000} has 209 digits.

Equation 2.7.1 is quite amazing: it isn't even obvious that the right side is an integer! The key to understanding it is the matrix equation

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix}. \qquad 2.7.2$$

The first equation says $a_n = a_n$, and the second says $a_{n+1} = a_n + a_{n-1}$. What have we gained? We see that

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2\begin{bmatrix} a_{n-2} \\ a_{n-1} \end{bmatrix} = \dots = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This looks useful, until you start computing the powers of the matrix, and discover that you are just computing Fibonacci numbers the old way. Is there a more effective way to compute the powers of a matrix?

Certainly there is an easy way to compute the powers of a *diagonal* matrix; you just raise all the diagonal entries to that power:

$$\begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_m \end{bmatrix}^n = \begin{bmatrix} c_1^n & 0 & \dots & 0 \\ 0 & c_2^n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_m^n \end{bmatrix}. \qquad 2.7.3$$

We will see that we can turn this to our advantage. Let

$$P = \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix}, \text{ so } P^{-1} = \frac{1}{4\sqrt{5}}\begin{bmatrix} \sqrt{5}-1 & 2 \\ \sqrt{5}+1 & -2 \end{bmatrix} \qquad 2.7.4$$

and "observe" that if we set $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, then

$$P^{-1}AP = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \text{ is diagonal.} \qquad 2.7.5$$

This has the following remarkable consequence:

$$(P^{-1}AP)^n = (P^{-1}A\underbrace{P)(P^{-1}}_{I}AP)\dots(P^{-1}A\underbrace{P)(P^{-1}}_{I}AP) = P^{-1}A^nP, \qquad 2.7.6$$