Appendix: Analysis

A.0 Introduction

This appendix is intended for students using this book for a class in analysis, and for the occasional student in a beginning course who has mastered the statement of the theorem and wishes to delve further.

In addition to proofs of statements not proved in the main text, it includes a justification of arithmetic (appendix A.1), a discussion of cubic and quartic equations (appendix A.2), the Heine-Borel theorem (appendix A.3), a definition of "big O" (appendix A.11), Stirling's formula (appendix A.16), a definition of Lebesgue measure (definition A22.5) and a discussion of what sets are measurable (theorem A22.6 and example A22.7), a new statement about k-dimensional volume 0 (appendix A.23), and a discussion of the pullback (appendix A.25).

Some proofs for chapter 2 use material from chapter 4.

A.1 Arithmetic of real numbers

It is harder than one might think to define arithmetic for the reals – addition, multiplication, subtraction, and division – and to show that the usual rules of arithmetic hold. Addition and multiplication as taught in elementary school always start at the right, and for reals there is no right.

Because you learned to add, subtract, divide, and multiply in elementary school, the algorithms used may seem obvious. But understanding how computers simulate real numbers is not nearly as routine as you might imagine. A real number involves an infinite amount of information, and computers cannot handle such things: they compute with finite decimals. This inevitably involves rounding off, and writing arithmetic subroutines that minimize round-off errors is a whole art in itself. In particular, computer addition and multiplication are not commutative or associative. Anyone who really wants to understand numerical problems has to take a serious interest in "computer arithmetic."

Recall that we defined the reals as "the set of infinite decimals." For rigor's sake we will now spell out exactly what this means; to avoid making special conventions, we will write our infinite decimals with leading 0's.

Definition A1.1 (Real numbers). The set of real numbers is the set of equivalence classes of expressions

$$\pm \dots 000 a_n a_{n-1} \dots a_0 \underset{\uparrow}{.} a_{-1} a_{-2} \dots, \qquad A1.1$$

where all a_i are in $\{0,1,2,3,4,5,6,7,8,9\}$, and the arrow points to the decimal point. Two such expressions

$$a = \pm \dots 0 a_n a_{n-1} \dots a_0 . a_{-1} \dots \text{ and } b = \pm \dots 0 b_m b_{m-1} \dots b_0 . b_{-1} \dots$$

are equivalent if and only if any of the following conditions is met:

1. They are equal.
2. All a_i and all b_i are 0, and the signs are opposite (this equivalence class is called 0).
3. a and b have the same sign; there exists k such that $a_k \neq 9$ and $a_{k-1} = a_{k-2} = \dots = 9$, and

$$b_j = a_j \quad \text{for } j > k,\ b_k = a_k + 1,\ b_{k-1} = b_{k-2} = \dots = 0.$$

Definition A1.2 (k-truncation). The *k-truncation* of a real number $a = \ldots 000a_n a_{n-1} \cdots a_0 \,.\, a_{-1}a_{-2} \ldots$ is the finite decimal

$$[a]_k = \ldots a_n \ldots a_k 000 \ldots \qquad A1.2$$

For instance, if $a = 21.3578$, then $[a]_{-2} = 21.35$.

The underlying idea is to show that if you take two reals, truncate (cut) them further and further to the right and add them (or multiply them, or subtract them, etc.) and look only at the digits to the left of any fixed position, the digits we see will not be affected by where the truncation takes place, once it is well beyond where we are looking. The problem with this is that it isn't quite true.

Example A1.3 (Addition). Consider adding the following two numbers:

$$\begin{matrix} .222222\ldots 222\ldots \\ .777777\ldots 778\ldots. \end{matrix} \qquad A1.3$$

The sum of the truncated numbers will be $.9999\ldots 9$ if we truncate before the position of the 8, and $1.0000\ldots 0$ if we truncate after the 8. So there cannot be any rule which says, "the 100th digit will stay the same if you truncate after the Nth digit, however large N is." The "carry" can come from arbitrarily far to the right.

If you insist on defining everything in terms of digits, it can be done but is quite involved. Even showing that addition is associative involves at least six different cases, and although none is hard, keeping straight what you are doing is quite delicate. Exercise A1.6 should give you enough of a taste of this approach. Proposition A1.6 allows a general treatment; the development is quite abstract.

Since we don't yet have a notion of subtraction in $\mathbb{R}$, we can't write $|x-y| < \epsilon$ for $x, y \in \mathbb{R}$, much less

$$\sum (x_i - y_i)^2 < \epsilon^2,$$

which involves addition and multiplication besides. Our definition of k-close uses only subtraction of finite decimals.

Let us denote by $\mathbb{D}$ the set of finite decimals.

Definition A1.4 (Finite decimal continuity). A map $f : \mathbb{D}^n \to \mathbb{D}$ is called *finite decimal continuous* (or $\mathbb{D}$-continuous) if for all integers N and k, there exists an integer l such that if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are two elements of $\mathbb{D}^n$ with all $|x_i|, |y_i| < N$, and if $|x_i - y_i| < 10^{-l}$ for all $i = 1, \ldots, n$, then

$$|f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n)| < 10^{-k}. \qquad A1.4$$

The letter A stands for addition, M for multiplication, and S for subtraction; the function Assoc is needed to prove associativity of addition.

Exercise A1.2 asks you to show that the functions $A(x,y) = x+y$, $M(x,y) = xy$, $S(x,y) = x-y$, and $\text{Assoc}(x,y,z) = (x+y)+z$ are $\mathbb{D}$-continuous and that $1/x$ is not.

To see why definition A1.4 is the right definition, we need to say what it means for two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ to be close.

For instance, if $a = 1.23000013$ and $b = 1.22999903$, then a and b are not 7-close, since

$$[a]_{-7} - [b]_{-7} = 11 \times 10^{-7} > 10^{-7}$$

but they are 6-close, since

$$[a]_{-6} - [b]_{-6} = 10^{-6}.$$

Definition A1.5 (k-close). Two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are *k-close* if for each $i = 1, \ldots, n$, then $\big|[x_i]_{-k} - [y_i]_{-k}\big| \le 10^{-k}$.

Notice that if two numbers are k-close for all k, then they are equal (see exercise A1.1).

If $f : \mathbb{D}^n \to \mathbb{D}$ is $\mathbb{D}$-continuous, then define $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ by the formula

$$\widetilde{f}(\mathbf{x}) = \sup_k \inf_{l \le -k} f([x_1]_l, \ldots, [x_n]_l). \qquad A1.5$$

The notion of k-close is the correct way of saying that two numbers agree to k digits after the decimal point. It takes into account the convention by which a number ending in all 9's is equal to the rounded up number ending in all 0's: the numbers .9998 and 1.0001 are 3-close.

Proposition A1.6. *The function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ is the unique function that coincides with f on $\mathbb{D}^n$ and which satisfies the continuity condition that for all $k \in \mathbb{N}$, for all $N \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that when $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are l-close and all coordinates x_i of $\mathbf{x}$ satisfy $|x_i| < N$, then $\widetilde{f}(\mathbf{x})$ and $\widetilde{f}(\mathbf{y})$ are k-close.*

The functions $\widetilde{A}$ and $\widetilde{M}$ satisfy the conditions of proposition A1.6; thus they apply to the real numbers, while A and M without tildes apply to finite decimals.

The proof is the object of exercise A1.4. With this proposition, setting up arithmetic for the reals is plain sailing.

Consider the $\mathbb{D}$-continuous functions $A(x, y) = x + y$ and $M(x, y) = xy$; then we define addition of reals by setting

$$x + y = \widetilde{A}(x, y) \quad \text{and} \quad xy = \widetilde{M}(x, y). \qquad A1.6$$

It isn't harder to show that the basic laws of arithmetic hold:

$x + y = y + x$	Addition is commutative.
$(x + y) + z = x + (y + z)$	Addition is associative.
$x + (-x) = 0$	Existence of additive inverse.
$xy = yx$	Multiplication is commutative.
$(xy)z = x(yz)$	Multiplication is associative.
$x(y + z) = xy + xz$	Multiplication is distributive over addition.

These are all proved the same way. Let us prove the last. Consider the function $\mathbb{D}^3 \to \mathbb{D}$ given by

$$F(x, y, z) = M\big(x, A(y, z)\big) - A\big(M(x, y), M(x, z)\big). \qquad A1.7$$

We leave it to you to check that F is $\mathbb{D}$-continuous, and that

$$\widetilde{F}(x, y, z) = \widetilde{M}\big(x, \widetilde{A}(y, z)\big) - \widetilde{A}\big(\widetilde{M}(x, y), \widetilde{M}(x, z)\big). \qquad A1.8$$

But F is identically 0 on $\mathbb{D}^3$, and the identically 0 function on $\mathbb{R}^3$ coincides with 0 on $\mathbb{D}^3$ and satisfies the continuity condition of proposition A1.6, so $\widetilde{F}$ vanishes identically by the uniqueness part of proposition A1.6. That is what was to be proved.

It is one of the basic irritants of elementary school math that division is not defined in the world of finite decimals.

This sets up almost all of arithmetic; the missing piece is division. Exercise A1.3 asks you to define division in the reals.

Exercises for section A.1

A1.1 Show that if two numbers are k-close for all k, then they are equal.

***A1.2** Show that the functions $A(x,y) = x+y$, $M(x,y) = xy$, $S(x,y) = x-y$, and $\mathrm{Assoc}(x,y,z) = (x+y)+z$ are $\mathbb{D}$-continuous, and that $1/x$ is not. Notice that for A and S, the l of definition A1.4 does not depend on N, but that for M, l does depend on N.

Asterisks (*) denote difficult exercises. Two stars indicate a particularly challenging exercise.

***A1.3** Define division of reals, using the following steps.

a. Show that the algorithm of long division of a positive finite decimal a by a positive finite decimal b defines a repeating decimal a/b, and that $b(a/b) = a$.

b. Show that the function $\mathrm{inv}(x)$ defined for $x > 0$ by the formula

$$\mathrm{inv}(x) = \inf_k \frac{1}{[x]_k}$$

satisfies $x\,\mathrm{inv}(x) = 1$ for all $x > 0$.

c. Define the inverse for any $x \neq 0$, and show that $x\,\mathrm{inv}(x) = 1$ for all $x \neq 0$.

FIGURE A1.1.

Giuseppe Peano (1858–1932) was the son of Italian farmers. He discovered Peano curves in 1890. He was noted for his rigor and his ability to disprove theorems by other mathematicians by finding exceptions. He also proposed a universal language with words from English, French, German, and Latin but no grammar.

****A1.4** Prove proposition A1.6. This can be broken into the following steps.

a. Show that $\sup_k \inf_{l \geq k} f([x_1]_l, \dots, [x_n]_l)$ is well defined (i.e., that the sets of numbers involved are bounded). Looking at the function S from exercise A1.2, explain why both the sup and the inf are there.

b. Show that the function $\widetilde{f}$ has the required continuity properties.

c. Show the uniqueness.

****A1.5** In this exercise we will construct a continuous map $\gamma : [0,1] \to \mathbb{R}^2$, the image of which is a (full) triangle T; such a mapping is called a *Peano curve.* We will write our numbers in $[0,1]$ in base 2, so such a number might be something like $.0011101000011\dots$, and we will use the table below

digit / position	0	1
even	left	right
odd	right	left

Exercise A1.5: Peano curves give onto, continuous mappings from $\mathbb{R} \to \mathbb{R}^2$. Analogues of Peano curves can be constructed from $\mathbb{R}$ to the infinite-dimensional vector space $\mathcal{C}[0,1]$!

Take a right triangle T. We will associate to a string $\underline{s} = s_1, s_2, \dots$ of digits 0 and 1 a sequence of points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots$ of T by starting at the point $\mathbf{x}_0$, dropping the perpendicular to the opposite side, landing at $\mathbf{x}_1$, and turning left or right according to the digit s_1, as interpreted by the bottom line of the table, since this digit is the first digit (and therefore in an odd position): on 0 turn right and on 1 turn left.

Now drop the perpendicular to the opposite side, landing at $\mathbf{x}_2$, and turn right or left according to the digit s_2, as interpreted by the top line of the table, etc.

This construction is illustrated in figure A1.5.

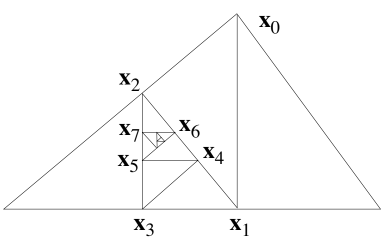

Figure for exercise A1.5

This sequence corresponds to the string of digits

$$00100010010\dots.$$

a. Show that for any string of digits $(\underline{s})$, the sequence $\mathbf{x}_n(\underline{s})$ converges.

b. Suppose $t \in [0,1]$ can be written in base 2 in two different ways (one ending in 0's and the other in 1's), and call $(\underline{s})$, $(\underline{s}')$ the two strings of digits. Show that

$$\lim_{n\to\infty} \mathbf{x}_n(\underline{s}) = \lim_{n\to\infty} \mathbf{x}_n(\underline{s}').$$

Hint: Construct the sequences associated to $.1000\dots$ and $.0111\dots$.

This allows us to define $\gamma(t) = \lim_{n\to\infty} \mathbf{x}_n(\underline{s})$.

c. Show that γ is continuous.